



Global Optimization for Structural Design by Generalized Benders' Decomposition

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Global Optimization for Structural Design by Generalized Benders' Decomposition

Eduardo Muñoz Queupumil

Ph.D. Thesis

DTU - June 2010

Global Optimization for
Structural Design
by
Generalized Benders'
Decomposition

by

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Title of the thesis:

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Preface

This thesis is submitted in partial fulfillment for obtaining the degree of Ph.D. of the Technical university of Denmark. The Ph.D. project has been carried out during the period June 1st 2007 – May 31st 2010. The research work has been carried out mainly in the Mathematics Department of the Technical University of Denmark. The project was funded by the Danish Research Council for Technology and Production Sciences (FTP). Part of the work was also carried out at the Structural Analysis Department of the Technical University of Munich. Associate Professor Mathias Stolpe and Professor Erik Lund were the supervisors of the project. I start by expressing that I am so grateful for my two supervisors, who have given me best conditions that allow me to be able to finish today my Ph.D thesis. I would like to thank Professor Martin P. Bendsøe for the supervision received during the one and a half year period he was co-supervisor in the project, and for being up to some point, the person who recommended me for doing a Ph.D in DTU. I am also very grateful to Professor Kai-Uwe Bletzinger from the Technical University of Munich for being a great host during the period I visited TUM. I would also like to express that I am very grateful for the incredibly good collaboration synergy and hard work sessions that I experienced collaborating with Christian G. Hvejsel from Aalborg University. The Mathematics Department has been a really nice place to work, and I am so happy for having spend three years of my life there. I want to thank my colleagues and all people of DTU-MAT.

Kgs. Lyngby, June 2010

Eduardo Munoz Queupumil

Resumé (in Danish)

Global optimering for strukturelt design med Generaliseret Benders' Dekomposition

Denne afhandling præsenterer anvendelsen af globale optimeringsmetoder til at designe optimale mekaniske strukturer. Hovedformålet med projektet er at undersøge nye modeller og metoder til at designe optimale kompositte, laminerede strukturer. Afhandlingen viser desuden udviklingen af en væsentlig forbedring i formuleringen af den Generaliserede Benders' Dekomposition (GBD) teknik for strukturelle optimeringsproblemer. Denne forbedrede teknik benyttes først til at designe optimale gitterstrukturer, hvilket vises med flere eksempler. Forbedringerne af denne metode udnyttes dernæst progressivt til alle de forskellige anvendelser, som er studeret i afhandlingen. Den udviklede GBD metode og algoritme benyttes til at løse minimum eftergivigheds- og minimum vægtproblemer. Den forbedrede metode gjorde det specielt muligt at løse simple minimum eftergivighedsproblemer for global optimalitet for numeriske eksempler med op til 23.000 designvariable med en lille numerisk tolerance. Metoden er generaliseret til at inkludere relevante lokale brudkriterier for kompositte, laminerede strukturer. Da lokale brudkriterier generelt ikke er konvekse, gør det opgaven med at bevise global optimalitet sværere og mere tidskrævende. Flere forskellige heuristiske metoder foreslås og benyttes til yderligere at forbedre GBD algoritmen. Disse heuristiske metoder testes, og det vises, at de har en væsentlig indflydelse på konvergensen af GBD metoden. Hovedbidraget til denne afhandling er at forbedre GBD algoritmen, som bliver brugt til at optimere strukturelle designproblemer. Disse forbedringer kan let generaliseres til mange forskellige konvekse problemstillinger i området indenfor mikset heltalsoptimering. Denne afhandling repræsenterer desuden det første arbejde med at løse for global optimalitet for strukturelle designproblemer, der inkluderer brudkriterier, og har numeriske eksempler med op til 400 designvariable.

Summary

Global Optimization for Structural Design by Generalized Benders' Decomposition

This thesis presents the application of global optimization methods to the design of optimal mechanical structures.

The main objective of the project is to investigate new models and methods for the design of optimal composite laminated structures. The thesis shows as well the development of a significant improvement in the formulation of the Generalized Benders' Decomposition (GBD) technique in structural optimization problems. This improved technique is used at first in the design of optimal truss structures, where several numerical examples are showed. Furthermore, the improvements in this method are used progressively along the thesis in all applications.

The developed GBD method and algorithm is applied in the resolution of minimum compliance and minimum weight problems. In particular, these improvements allowed us to solve simple minimum compliance design problems to global optimality in numerical examples of up to 23.000 design variables, with a quite small numerical tolerance. The method is generalized for the inclusion of relevant local failure criteria for composite laminated structures. In general, local failure criteria functions are non convex, which makes the task of proving global optimality even more difficult and time consuming.

Besides, several heuristics have been proposed and used to improve the performance of the already improved GBD algorithm. These heuristics are tested and proved to be of significant impact in the convergence of the GBD method.

The main contribution of this thesis is to provide a significant improvement in the GBD algorithm, which are applied in structural design problems. These improvements can be easily generalized to all kinds of convex problems in the mixed integer optimization field. Besides, this thesis represents the first work addressing the resolution to global optimality of structural design problems including local failure criteria, and having numerical examples of structural design problems of up to 400 design variables.

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Introduction

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The main subject of this thesis is the optimal design of mechanical structures. More specifically, this thesis aims to develop models, methods and algorithms to design optimal composite laminated structures by means of mathematical programming. We use a standard method for mixed integer optimization called *Generalized Benders' Decomposition* (GBD, [13]) to attack the considered optimization models. Composite laminates are high technology manufacture structures, providing efficiency in the physical properties of the structure, while keeping the overall weight at a low level. This type of structures has become popular in the 20th century, principally in the aeronautics industry, but also in many other areas of the engineering industry. As a matter of fact, this thesis has been inspired by the wind turbine blade design process, which includes a significant amount of composite laminated structures as a requirement in the design process.

The methods we have developed consider several design criteria in the optimization models. This thesis considers in general, the situation of solving these design optimization models to global optimality. Among the important design criteria we consider we can mention the global stiffness (compliance), weight and specific local failure criteria.

This thesis consists in a 8 chapters. Chapter one is the introduction the thesis, chapter two shows the resume of the articles included in the thesis. Chapter three to seven consist of five articles being the main work of this thesis. Articles 1, 2 and 3 (chapters 3, 4 and 5 respectively) deal with the theory of the GBD method applied to general topology optimization problems, and numerical applications in truss topology design problems. Articles 4 and 5 address the concrete problem of designing composite laminated structures.

In this introduction, we set a common nomenclature to be used throughout the thesis. Nevertheless, we do not use exactly the same notation for all of the articles. In exchange, the notation for each article is adapted to the specific application that is being addressed. In general, the notation is clearly described in the problem formulation of each article. For the understanding of this thesis, we assume a basic knowledge in optimization, finite element method theory, statics, and material sciences. In the following, a brief introduction will help to understand the necessary basic concepts, models and methods described in the thesis.

1.1 Preliminaries

The optimization design problems we consider need the understanding of a few important concepts, which are briefly described here. Since in this thesis the use of optimal design models is applied to structural design problems, we use the words design and structure in the same sense. Another important consideration is that since the optimization models used in this thesis are applied to optimal design, the use of the word *design variable* often replaces the standard nomenclature of *decision variables* of the Optimization field (even though in the models we present, other physical quantities are involved in the optimization problem, they are not considered truly decision variables, since they are restricted to satisfy the governing equations).

Consider a domain $\Omega_c \in \mathbb{R}^2$ or \mathbb{R}^3 , with piece-wise differentiable boundary. Ω_c is called the *continuous design domain*, and define the physical limits in which the design problem will be formulated. Suppose now the existence of a set of loads (external static forces) $f_1(w), \dots, f_m(w) \in \mathbb{F}(\Omega_c)$, where $\mathbb{F}(\Omega_c)$ represents the set of admissible static loads (volume, surface and point forces) in Ω_c . In order to have a static mechanical model, suitable support (or boundary conditions) conditions need to be imposed. To simplify the modeling, we suppose that these support conditions fix the design in a subset Ω_{fix} of the boundary of the design space (i.e., $\Omega_{fix} \subset \partial\Omega_c$).

There are many ways to define what a design is, for practical or abstract mathematical purposes. In this thesis, the concept of design will be consider as any mathematical object (element in a finite or infinite dimensional vector space) that defines uniquely the physical representation of a design through a mathematical model. This definition is in fact vague, but the relation between a design as a real object and a mathematical representation is not trivial from the abstract point of

view. So for our purposes a design will be accepted as any function of the type

$$\begin{aligned} D : \Omega_c &\longrightarrow \mathbb{E} \\ \omega &\longrightarrow (pr_1(w), \dots, pr_q(w)), \end{aligned}$$

where \mathbb{E} represents the space of physical properties defining the given design.

At the first level, we can for example define a design by the distribution of any set of material properties throughout the design domain (mass density, stiffness, etc).

With respect to the mechanical model, we suppose (for simplicity) that only surface or punctual loads are applied over the boundary of the design domain $\partial\Omega_c$. For a given design, the static equations, or continuous equilibrium equations, relating the stress tensor function $\sigma_l(w)$ for each load f_l are given by

$$\begin{aligned} \nabla \cdot \sigma_l(w) &= 0, \quad \text{in } \Omega_c, \\ \sigma_l(w) \cdot \hat{n} &= f_l(w), \quad \text{in } \partial\Omega_c, \end{aligned} \tag{1.1}$$

where \hat{n} represents a unit vector in the normal direction of the boundary $\partial\Omega_c$. The stress is a physical quantity that could be defined as “the intensity of the internal forces acting between particles of a deformable body across any given plane” (see for example [3], or any basic text in continuum mechanics).

Equation (1.1) represents the governing equations of the design problem, and it represents the strong formulation of the static problem. The corresponding weak formulation of (1.1) can be obtained for example from the *virtual work principle*, and it is given by

$$-\int_{\Omega_c} \delta \varepsilon_{ij}(w) \sigma_{ij}(w) dV + \int_{\partial\Omega_c} \delta u_j(w) f_j dS = 0, \quad \forall u_j \text{ admissible.} \tag{1.2}$$

Here, $\varepsilon = \{\varepsilon_{ij}\}$ represent the *Cauchy's strain tensor* (see for instance [22], or any text in basic continuum mechanics). Through this thesis, the Cauchy's strain tensor will be just called *strain tensor*. The strain tensor is a physical measure of the deformation of a continuum body and it is defined as

$$\varepsilon(w) = \frac{1}{2}(\nabla u(w)^T + \nabla u(w)).$$

For the scope of this thesis, linear elasticity is considered. This means that the stress tensor and the strain tensor are related through Hook's law

$$\sigma(w) = E(w)\varepsilon(w). \tag{1.3}$$

The weak formulation (1.2) is used to build the finite element model through a finite element discretization. This discretization means that the continuous domain Ω_c is replaced by a discretized design space Ω_d , composed of n finite elements.

There are several discretization methods to treat weak formulations of partial differential equations. Among them we can mention the Finite Element Method

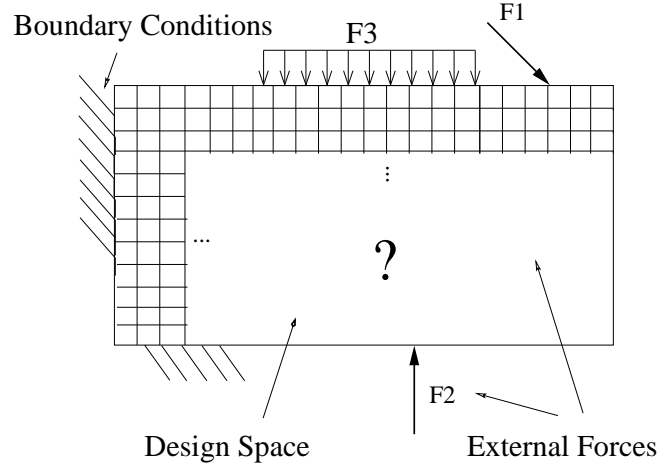


Figure 1.1: Representation of the discretized design space, boundary conditions and external loads related to a structural design problem

(FEM, or FE method) discretization (see for instance [20, 9, 4]), the Finite Volume Method (FVM) discretization (see for example [11, 24]), and the finite difference discretization ([19, 4]). The most popular among these within the field of structural and mechanical analysis is the FE method. In the last five years a new variation of the finite element method, called isogeometric analysis method (references) has become quite popular, and has already shown promising results. It is possible that in the close future, this new analysis technique will take over the finite element method, at least in the field of structural analysis. In this thesis, all discretized models have been built by standard FE method (shell elements discretization in the case of laminated structures). The correctness of the considered analysis models is not questioned, and they are not the subject of this thesis. The reader can refer to any text in finite element methods for further details of this method.

The discretization of the design space supposes as well the discretization of all quantities involved in the governing equations, such as external loads, displacements and boundary conditions of the problem. A simple example of a 2 – D discretized domain and boundary conditions is illustrated in figure 1.1.

The finite element formulation of the equilibrium equations, related to the strong and weak formulations (1.1) and (1.2) is given by

$$K(x)u_l = f_l, \quad l = 1, \dots, m, \quad (1.4)$$

where $K(x) \in \mathbb{R}^{d \times d}$ is called the stiffness matrix of the FE model, $u_l \in \mathbb{R}^d$ is the discretized displacement, and $f_l \in \mathbb{R}^d \setminus \{0\}$ is the discretized external load related to the continuous load distribution $f_l(w) \in \mathbb{F}(\Omega_c)$. In all articles being part of this thesis, the equilibrium equations are formulated in the FE form (1.4).

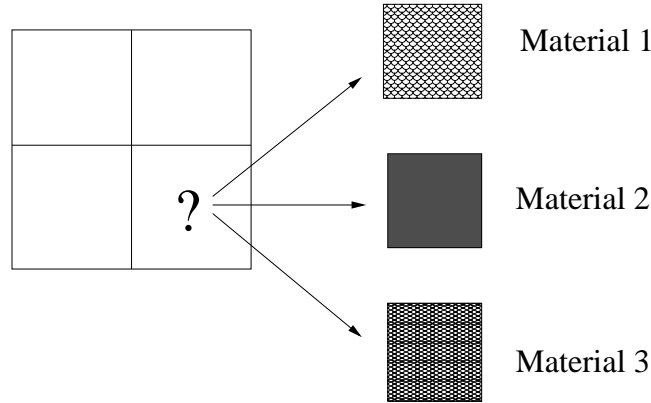


Figure 1.2: Discrete material optimization approach. For each design element, the selection among several candidate material has to be done.

1.1.1 Free Material Optimization Approach

If the design is defined by its stress-strain tensor $E(w)$, $w \in \Omega_c$, and the space of all possible tensor is defined as the natural design space, the optimal design formulations obtaining through this modeling is called *Free Material Optimization* (FMO, see [16]). One of the consequences of discretizing the design space is that the material properties tensor $E(w)$ is replaced by a discretized tensor $E_d(t)$, $t \in \Omega_d = \{0, 1\}^n$, where n is the dimensional size of the design space. The discretization simplifies the size of the material properties tensor, taking it from an infinite dimensional function space, to a finite dimensional vector space. The FMO approach is able to formulate problems to obtain the best distribution of the material tensor properties in Ω_d . In addition, FMO allows to establish well formulated problems, with strong mathematical properties, such as convexity properties, necessary for convergence to global optimality. The FMO field has shown a considerable contribution to the structural design field. However, this approach supposes that the space of allowed material properties is larger than the set of existing materials in nature. Therefore, it is possible that the results obtained by the FMO approach do not give realizable designs. This is currently one of the main research lines in FMO nowadays.

1.1.2 Discrete Material Optimization Approach

The approach we choose to use instead, is the so called *Discrete Material Optimization*, DMO (stegmann-lund2005). In this approach, we consider for each design element, a given finite set of n^c candidate materials, each one described by a specific material properties (strain-stress tensor) E_1, \dots, E_{n^c} . The situation is represented in figure 1.2, where the choice among three candidate materials has to be done.

This approach supposes that a design is defined as a sequence $E(t_1), \dots, E(t_n)$ of n strain-stress tensors, one per design element. This means that for the j -th design element, the strain-stress tensor is given by the conditional function

$$E(t_j) = \begin{cases} E_1, & \text{if Material 1 is chosen,} \\ \vdots & \\ E_{n_j^c}, & \text{if Material } n_j^c \text{ is chosen.} \end{cases}$$

This condition for the selection of the material can also be modeled with binary variables by introducing a double index variable $x_{ij} \in \{0, 1\}$, for each material i and for each design element j .

$$x_{ij} = \begin{cases} 1, & \text{if material } i \text{ is chosen for the element } j. \\ 0, & \text{otherwise.} \end{cases} \quad (1.5)$$

$x = \{x_{ij}\}_{ij}$ is known throughout this thesis, as the *design variable vector* or simply *design variable*. Note that the design variable vector x is a vector in $\{0, 1\}^{\tilde{n}}$, with $\tilde{n} = n^c \cdot n$ and the double index should not be confused with matrix coefficients representation. This definition of the design variables needs the inclusion of a selection condition to be valid. This selection condition supposes that among all available candidate materials in a design element, only one of them must be chosen. This condition is represented by the selection material constraint

$$\sum_{i=1}^{n^c} x_{ij} = 1, \quad \forall j = 1, \dots, n.$$

In the particular case where one of the material candidates is void, the DMO problem becomes a topology optimization problem (see [6]). This means that in a sense, the DMO is in fact, a generalization of the topology optimization approach.

1.2 Material Properties

When formulating the DMO problem, the pre-selection of the set of candidate materials to be included in the optimization problems is quite important for the results of the optimization process. For example, it is clear that if the selection is done among materials with too similar characteristics, the optimization process may be worthless. Or, if two materials with similar characteristics but one of them with better material properties, the optimization process will always choose the material with better characteristics (unless price constraints are considered). Therefore it is important to preselect an heterogeneous set of material candidates, where a trade-off between the different strength of the candidates has to be done (normal stiffness, shear stiffness, mass density, price, etc.). In this thesis, two types of candidate materials are considered. First, we consider *Isotropic materials*, which are defined as materials having the same mechanical behavior independently of the direction of the local stresses. The second type of candidate material we consider is the class of *Orthotropic Materials*, defined as materials with different mechanical behavior, depending on the orientation of the local stresses, but their mechanical behavior

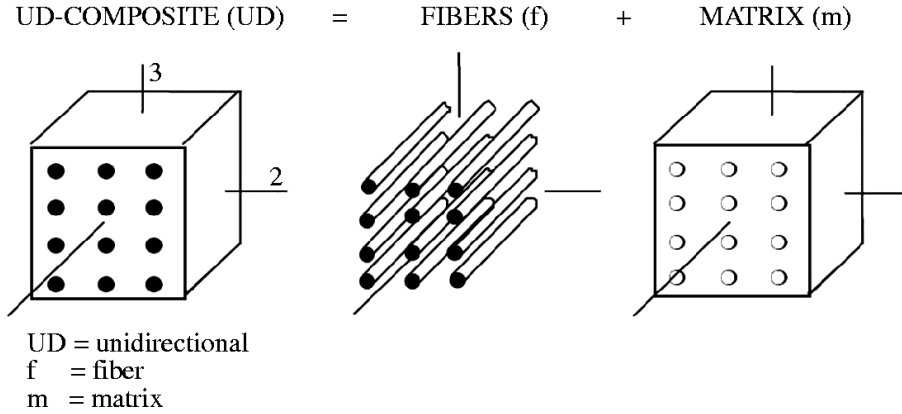


Figure 1.3: Graphical representation of a composite material composed of unidirectional fiber and a matrix material. This type of material is usually used in the design of composite laminated structures.

can be computed at any orientation by using a rotation matrix with respect to the principal axes of the material coordinate system. Orthotropic materials are a particular case of anisotropic materials, for which this rotational relationship between the mechanical behavior with respect to the orientation does not necessarily exists.

1.3 Composite Laminated Structures

The main application of this thesis concerns the design of optimal composite laminated structures. Therefore we introduce here briefly this type of structures.

First we start by defining the concept of composite material. *Composite materials* refers to “two or more materials combined on a macroscopic scale to form a useful third material” ([15]). This definition emphasizes that the combination of the materials must be done at a macroscopic scale, i.e., the material is composed of two or more faces, which can be distinguished by naked eye. It also means that materials combined at a microscopic scale (i.e, no faces can be distinguished by naked eye) are not composite materials (for instance, alloying of metals). Composite materials can be classified in four different groups, depending on the nature of the mixture forming the material. For the scope of this thesis, we consider composite materials in the sense of materials composed of fibers of materials embedded in a matrix (see Figure 1.3).

The matrix material can be of any nature, such as organic, metallic, ceramic or carbon. The correct combination of the matrix material and the fiber material improves the mechanical properties of the resulting composite material in comparison with the material in bulk form. The matrix material gives support to the fibers, and it allows a better distribution of the fibers, minimizing at the same time, the presence of imperfections in the material. Besides, it is a mean for the transmission of forces among the fibers. In most of the cases, this type of composite materials behaves as orthotropic materials.

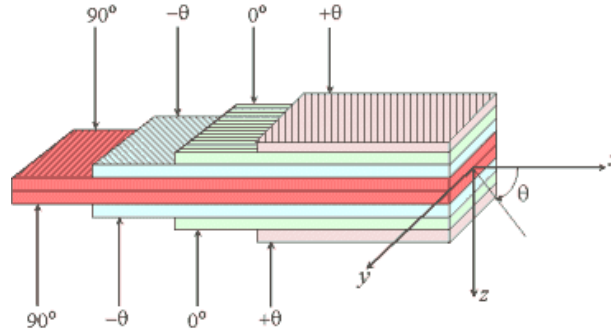


Figure 1.4: Graphical representation of the lay-out of a composite laminated structure. The design problem consists in choosing the material to be used in each layer, and the angle orientation in the case of unidirectional composite materials.

Composite laminated structures refers to structures composed of “layers of at least two different materials that are bonded together” ([15]). The layers are stacked together through the matrix material (see Figure 1.3).

One of the most convenient ways to produce high performance structures is to design composite laminated structures, where the layers are composed of composite materials and possible other types of materials. If on top of this, we use optimization techniques for the design of this type of structures, we can improve substantially the physical response of the structures. For example, we can improve stiffness, strength, weight, corrosion resistance, wear resistance, acoustic insulation, thermal insulation, etc. These properties can of course not be improved all of them at the same time. This is not really a problem, since it is very unlikely to require a structure to satisfy all these properties simultaneously.

The design of composite laminate structures through optimization techniques may have a considerable impact in the performance of the structures. By using the DMO approach, we can develop optimization models to determine the best distribution of a set of candidate materials being part of the composite laminated structure. If we also consider angle orientation, we can develop optimization models to decide the optimal distribution of the material and angle orientation of the material in the structure. Note that the formulation of the optimization model is one thing, and to solve it is a complete different problem. This thesis deals with developing optimization models and methods capable of solving these models for the design of optimal composite laminated structures.

1.4 Relevant Criteria for the Design Problem

In this thesis we deal with a number of global and local criteria for the design of optimal structures. Among the global criteria considered, we can mention the compliance or the weight, while about local criteria, we can mention local failure criteria. These functions will be briefly described in the next subsections.

1.4.1 Mass

The mass is one of the most important criteria in structural optimization. Designing the lightest structure satisfying all the structural requirements is often the goal in many applications. If $\rho(w)$ is the density distribution of a given design in the continuous design space Ω_c , its mass is computed by.

$$M(\rho) = \int_{\Omega_c} \rho^T(\omega) d\omega.$$

Considering the finite and design element discretization, $\rho \in \mathbb{R}^{\tilde{n}}$ is the vector of densities related to the design variable $x \in \{0,1\}^{\tilde{n}}$ (given by (1.5)). Then, the corresponding mass is computed as the product

$$\rho^T x.$$

This form for the mass of designs is used through the thesis.

1.4.2 Compliance

The compliance is a measure of the global stiffness of the structure. More specifically, it measures the deformation work made by an external load. It is also one of the most relevant criteria in the structural optimization field, mainly because of its strong mathematical properties (convexity, auto-adjointness), that simplifies the analysis and optimization process. The compliance $c_l(u)$ due to the external load $f_l(w)$ is given by

$$c_l(u) = \int_{\Omega_c} f_l(w)^T u_l(w) d\omega_c.$$

Considering the finite and design element discretization, the discrete compliance can be expressed as

$$c_l(u_l) = f_l^T u_l,$$

If the equilibrium equations (1.4) have a unique solution, the displacement field u_l is a function of the design variables, i.e., $u_l = u_l(x)$. where $u_l \in \mathbb{R}^d$ is the solution (if this exists) to the equilibrium equations (1.4). Since the equilibrium equations relate the design variable x with the displacement field u_l , the displacement field depends on the design variable $u_l = u_l(x)$. This dependency allows us to write the discrete compliance as a function on the design variable x

$$c_l(x) = f_l^T u_l(x).$$

This is the form for the compliance constraint that is used in this thesis.

1.4.3 Additional Linear Constraints

We include this item to cover other potential interesting criteria to be included in the model. For example, cost functions may have a major importance in practical applications. However, since from the mathematical point of view, they are weight-type criteria, we did not explicitly included them in any of the formulations of this thesis. Nevertheless, it is worth to mention it, to motivate the application of these methods in real design problems. Other relevant constraints correspond to linear design constraints, which may derive from technical requirements to be satisfied by the out coming design. If $A \in \mathbb{R}^{p \times n}$ is a matrix, and $b \in \mathbb{R}^p$ is a given vector, then the p additional linear constraints can be represented by a linear constraint of the type

$$A \cdot x \leq b.$$

1.4.4 Local Failure Criteria in Composite Laminated Structures

By local failure criteria we mean the design restrictions based on local information of a structure, which is able to predict (hopefully in a conservative way) the local failure (fracture, delamination, fatigue, etc.) of the structure. We represent immediately the local failure function in the finite/design element discretization. Then, the local failure criteria takes the general form

$$F(x, u_l) \leq 0,$$

where F is in general, a non linear vectorial function in its domain

$$\begin{aligned} F : \mathbb{R}^{\tilde{n}} \times \mathbb{R}^d &\longrightarrow \mathbb{R}^q \\ F(x, u) &\longrightarrow (F_1((x, u)), \dots, F_q(x, u)) \end{aligned}$$

$F(x, u)$ is in most of the cases a function on the strain or stress tensor (since these can be modeled as functions in x and u), but this is not the only possibility. For example, if $F_j(x, u) = \|\sigma_j(x, u)\|_\infty$, $j = 1, \dots, n$, the local failure criteria considered is the maximum strain criterion. If $F_j(x, u) = \|\varepsilon_j(x, u)\|_\infty$, $j = 1, \dots, n$, we are talking about the maximum stress criterion. The Tsai-Wu and Tsai-Hill criteria are second degree polynomial functions in the stress tensor. We can find a large number of local failure criteria in the literature. For isotropic material, a number of failure criteria are used with great precision and correspondence with experiments (Von Misses, Tresca, etc., not treated in this thesis). On the other hand, for composite laminated structures, there is not a clear decision with respect to the most convenient local failure criteria to be considered, nor this thesis addresses this issue. Along this thesis, four local failure criteria are studied; The maximum strain criterion, the maximum stress criterion, the Tsai-Wu and the Tsai-Hill criteria. But, as it will be stated in the conclusion, the method developed in this thesis is able to handle any type of local failure design restrictions.

1.5 Relevant Optimization Models

As it was pointed out in the previous section, the topology optimization formulation is a particular case of the multimaterial DMO formulation. This means that we can state the general case of the DMO problem, without excluding the topology optimization case. The modeling of the DMO problem, combined with the FE discretization, and the relevant functions introduced in the previous section, allows us to formulate the design problems treated in this thesis. These are mainly two formulations. The first formulation is the minimum compliance problem, subjected to mass, local failure, and linear design constraints

$$\begin{aligned}
& \underset{\substack{x \in \mathbb{R}^{\tilde{n}} \\ u_1, \dots, u_m \in \mathbb{R}^d}}{\text{minimize}} && \max_{1 \leq l \leq m} \{f_l^T u_l\} \\
& \text{subject to} && K(x)u_l = f_l, \quad l = 1, \dots, m, \\
& && \rho^T x \leq M, \\
& && \sum_{i=1}^{n^c} x_{ij} = 1, \quad j = 1, \dots, n, \\
& && F(x, u_l) \leq 0, \quad l = 1, \dots, m, \\
& && Ax \leq b, \\
& && x \in \{0, 1\}^{\tilde{n}},
\end{aligned} \tag{1.6}$$

where M is the mass limit for the design of the structure. The second design problem considered in this thesis is the minimum weight problem, subjected to compliance, local failure, and linear design constraints

$$\begin{aligned}
& \underset{\substack{x \in \mathbb{R}^{\tilde{n}} \\ u_1, \dots, u_m \in \mathbb{R}^d}}{\text{minimize}} && \rho^T x \\
& \text{subject to} && K(x)u_l = f_l, \quad l = 1, \dots, m, \\
& && f_l^T u_l \leq C_l, \quad l = 1, \dots, m, \\
& && \sum_{i=1}^{n^c} x_{ij} = 1, \quad j = 1, \dots, n, \\
& && F(x, u_l) \leq 0, \quad l = 1, \dots, m, \\
& && Ax \leq b, \\
& && x \in \{0, 1\}^{\tilde{n}},
\end{aligned} \tag{1.7}$$

where $C_l > 0$ is the limit for the compliance value. In each chapter of this thesis, the necessary assumptions are stated clearly in the problem formulation section. In articles 1, 2, and 3, we study problems (1.6) and (1.7) applied to topology optimization problems (including perfect approximated void in truss topology problems) without failure criteria (i.e., excluding the condition $F(x, u) \leq 0$ from the model). Article 4 considers Problem (1.6) applied to the design of composite laminated structures without local failure criteria, and article 5 considers the complete version of these two problems.

1.6 Global Optimization

In this Thesis, the ultimate goal is to solve problems (1.6) and (1.7) to global optimality. This means that we want to find a couple $(x^*, u^*) \in \mathbb{R}^{\tilde{n}} \times \mathbb{R}^d$ being a global optimal solution of problems (1.6) or (1.7). Global optimization requires an extra effort when treating optimization problems with respect to standard optimization methods. While the latter are only focused in the search of locally optimal solutions, global optimization methods are focused in first, finding optimal solutions, and second, proving the non existence of better solutions in the entire design space. For some unconstrained optimization problems, finding a global optimal solution can be done in few steps, and maybe even analytically (for instance for smooth bounded analytical functions with a few local minima, where one of them is necessarily a global optimum). For other optimization problems this task may be practically impossible. The state of the art is that only a small fraction of the problems treated with standard optimization methods can be treated successfully with global optimization methods. The difficulty of this task depends basically in how easily we can discard the existence of better solutions in the design space. This can be done automatically when additional mathematical properties are satisfied. One of this properties is convexity of the optimization program. The reason is that for convex optimization problems, local optimality and global optimality are equivalent concepts. Another important conditions for global optimization are the strong duality properties (see, for example [12]). These properties allow us to solve an alternative linear problem to optimality (the so called dual optimization problem), and by this mean, to find valid lower bounds of the global optimal objective value for the original optimization problem. If the objective value of a design found is equal to the lower bound found by solving the dual problem, then both, the primal and the dual problem were solved to global optimality.

The following results are of great importance in global optimization, and in particular for this thesis.

1.6.1 Elements of Convex Optimization

For a deeper understanding in convex optimization, see [8]. Consider the generic minimization problem

$$\begin{aligned} & \underset{x \in \mathbb{R}^{\tilde{n}}}{\text{minimize}} && f_0(x) \\ & \text{subject to} && f_i(x) \leq b_i, i = 1, \dots, m, \end{aligned} \tag{1.8}$$

where $f_i : \mathbb{R}^{\tilde{n}} \rightarrow \mathbb{R}$, $i = 1, \dots, m$. We define here the concept of a *convex optimization problem*, which is used many times along the thesis. We begin by defining the concept of a convex function.

Definition 1. *The function $f_i(x)$ is said to be convex, if it satisfies*

$$f_i(\alpha x + (1 - \alpha)y) \leq \alpha f_i(x) + (1 - \alpha)f_i(y)$$

for all $x, y \in \mathbb{R}^n$, and for all $\alpha \in [0, 1]$.

If the functions $f_i(x)$ are of class $\mathcal{C}^1(\mathbb{R}^n)$, they necessarily satisfy

$$f(y) \geq f(x) + \nabla f(x)^T(y - x), \quad (1.9)$$

for all $x, y \in \mathbb{R}^n$.

Definition 2. *The problem (1.8) is said to be a convex problem if all functions $f_i(x)$, $i = 1, \dots, m$, are convex.*

These concepts are important for the results achieved in this thesis. In particular, the theoretical convergence to global optimality of the GBD algorithm applied to problems (1.6) or (1.7) is based on the convexity of some reformulations-relaxations of these two problems.

1.7 Mixed-Integer Optimization

Problems (1.6) and (1.7) correspond to non linear mixed 0 – 1 programs. A *Mixed 0-1 program* is an optimization program, where some of the decision variables (the design variables) can only take the values 0 or 1. The class of mixed 0 – 1 programs is a subset of the class of mixed-integer programs. *Mixed Integer Optimization* is the sub-area of the optimization field developing the theory and methods for solving mixed integer programs. This class of problems is also subdivided in the class of the linear mixed integer problems, and the class of the non linear mixed integer problems.

In general, solving a mixed integer problem is a quite difficult task. In exchange, when it a mixed integer problem is solved, it is solved to global optimality. This is due to the fact that in mixed integer programming, the concept of local optimum does not apply.

Among the most popular methods for solving mixed integer programs, we can mention the branch-and-bound methods ([7], [1, 2], [14], [17], and [25]), the branch-and-cut methods ([23]), the Outer Approximation method ([10]), and the Generalized Benders' Decomposition ([13, 21, 18, 5]).

1.7.1 Generalized Benders' Decomposition

As it was already mentioned, in this thesis we used the Generalized Benders' Decomposition method (for a detailed explanation of the method, see [13]) to treat problems (1.6) and (1.7). The principle of the method is briefly explained in each of the articles of this thesis. The method consists in replacing the original mixed 0 – 1 problem by two sequences of simpler optimization programs. The first sequence of programs considers only the design variable (plus an additional scalar continuous variable). This sequence is called the sequence of *master problems* (or sequence of *relaxed master problems*). These master problems intend to approximate the projection of the considered non linear function in the space of 0 – 1 variables. This approximation is made by linear functions added iteratively as constraints in the

master problems. These linear constraints are called *GBD cuts*. Figure 1.7.1 shows a representation of the approximation of an objective function by GBD cuts. From this figure it is also possible to see that the validity of the method should rely on some convexity assumptions. This fact is formally stated and proven in the thesis (for the case of problems (1.6) and (1.7)). The GBD cuts can be used to approximate a non linear objective function, in which case they are called *GBD optimality cuts*, or simply *optimality cuts*. Otherwise, they could be used to approximate a non linear constraint, in which case they are called *GBD feasibility cuts*, or simply *feasibility cuts* (only if no confusion could be done with other types of feasibility cut). In the case of problems (1.6) and (1.7), and through this thesis, the non linear function considered is the compliance function (1.4.2), so the GBD cuts in this context could also be called *compliance cuts*. The coefficients of the GBD cuts (i.e., the coefficients determining the GBD cut) are obtained by solving the second sequence of problems of the method. This sequence of problems considers only the continuous variables (in the case of problems (1.6) and (1.7), it corresponds to the displacement field $u \in \mathbb{R}^d$). This sequence is called the sequence of subproblems, and in the case of problems (1.6) and (1.7), they correspond to the analysis problem (1.4). In general, the GBD method uses the dual optimization information of the subproblem (1.4) to obtain the coefficients of the GBD cuts. In the thesis it was also proven that a suitable reformulation of the subproblem (1.4) is required to have the theory behind the GBD method to hold. The method aims to get first order approximations of the compliance function in the projected space (the space of 0 – 1 variables). This could be done easily by simple differentiation, if we could obtain the projection of the non linear functions in the space of 0 – 1 variables. Since this is not always possible, the GBD considers the use of dual optimization information, by computing the optimal Lagrange multipliers of the subproblems. These optimal Lagrange multipliers are used to compute the value of the gradients of the functions in the projected space, with no need to compute them explicitly analytically or by finite differences.

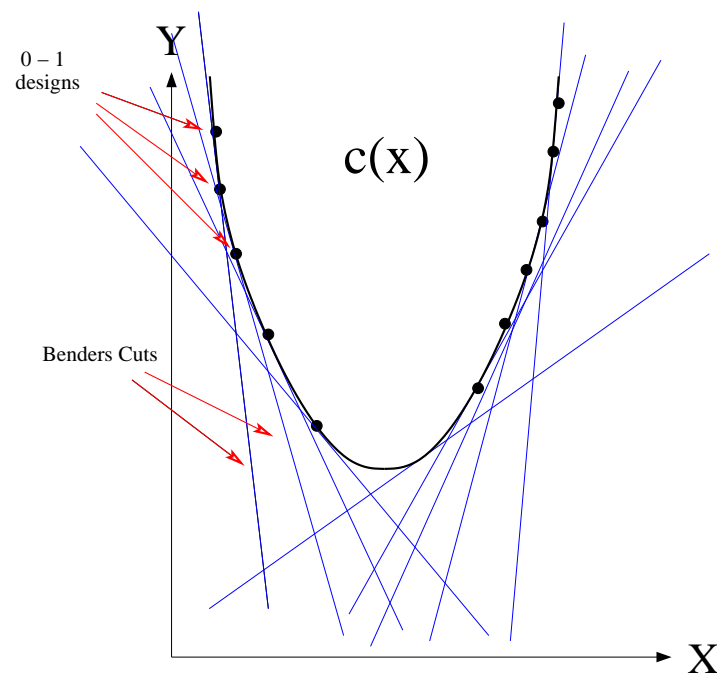


Figure 1.5: Graphical representation of the classical GBD method for approximating a convex objective function.

Bibliography

- [1] W. Achziger and M. Stolpe. Global optimization of truss topology with discrete bar areas - Part I: Theory of relaxed problems. *Computational Optimization and Applications*, (40):247–280, 2007. [13](#)
- [2] W. Achziger and M. Stolpe. Truss topology optimization with discrete design variables - guaranteed global optimality and benchmark examples. *Structural and Multidisciplinary Optimization*, 34(1):1–20, 2007. [13](#)
- [3] T.M. Atanackovic and A. Guran. *Theory of elasticity for scientists and engineers*. Birkhauser, 2000. [3](#)
- [4] K. Atkinson and W. Han. *Theoretical numerical analysis: A functional analysis framework*. Springer London, Limited, 2009. [4](#)
- [5] MJ Bagajewicz and V. Manousiouthakis. On the generalized Benders decomposition. *Computers & Chemical Engineering*, 15(10):691–700, 1991. [13](#)
- [6] M.P. Bendsøe and O. Sigmund. *Topology Optimization: Theory, Methods, and Applications*. Springer, 2003. [6](#)
- [7] S. Bollapragada, O. Ghattas, and J.N. Hooker. Optimal design of truss structures by logic-based branch and cut. *Operations Research*, 49(1):42–51, 2001. [13](#)
- [8] S.P. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge Univ Pr, 2004. [12](#)
- [9] S.C. Brenner and L.R. Scott. *The mathematical theory of finite element methods*. Springer, 1994. [4](#)
- [10] M.A. Duran and I.E. Grossmann. An outer-approximation algorithm for a class of mixed-integer nonlinear programs. *Mathematical Programming*, 36(3):307–339, 1986. [13](#)
- [11] R. Eymard, T. Gallouët, and R. Herbin. Finite volume methods. *Handbook of numerical analysis*, 7:713–1018, 2000. [4](#)
- [12] A.M. Geoffrion. Duality in non linear programming: a simplified applications-oriented development. *SIAM Review*, 13(1):1–37, 1971. [12](#)
- [13] A.M. Geoffrion. Generalized Benders decomposition. *Journal of Optimization Theory and Applications*, 10(4):237–260, 1972. [1](#), [13](#)
- [14] O.K. Gupta and A. Ravindran. Branch and bound experiments in convex nonlinear integer programming. *Management Science*, pages 1533–1546, 1985. [13](#)

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- [15] R.M. Jones. *Mechanics of Composite Materials*. Taylor & Francis, Inc., Philadelphia, 1999. 7, 8
 - [16] M. Kocvara, M. Stingl, and J. Zowe. Free material optimization: recent progress. *Optimization*, 57(1):79–100, 2008. 5
 - [17] A.H. Land and A.G. Doig. An automatic method of solving discrete programming problems. *Econometrica: Journal of the Econometric Society*, 28(3):497–520, 1960. 13
 - [18] R. Lazimy. Extension of the generalized Benders decomposition. *Communications in Applied Numerical Methods*, 6:195–203, 1986. 13
 - [19] AR Mitchell and DF Griffiths. The finite difference method in partial differential equations(Book). *Chichester, Sussex, England and New York, Wiley-Interscience, 1980. 281 p*, 1980. 4
 - [20] M. E. Plesha R. Cook, D. Malkus and R. J. Witt. *Concepts and Applications of Finite element Analysis*. John Wiley and Sons, 4th edition edition, 2002. 4
 - [21] N.V. Sahinidis and I.E. Grossman. Convergence properties of generalized Benders decomposition. *Computers and Chemical Engineering*, 15:481–491, 1991. 13
 - [22] J. SALENCON. Mécanique des milieux continus Tome 1: concepts généraux (avec CD-ROM). 2005. 3
 - [23] R.A. Stubbs and S. Mehrotra. Generating convex polynomial inequalities for mixed 0–1 programs. *Journal of Global Optimization*, 24(3):311–332, 2002. 13
 - [24] H.K. Versteeg and W. Malalasekera. *An introduction to computational fluid dynamics: the finite volume method*. Prentice Hall, 2007. 4
 - [25] L.A. Wolsey and G.L. Nemhauser. *Integer and combinatorial optimization*. Wiley-Interscience, 1998. 13

Summary of Results

This chapter describes briefly the contents of the five articles that are part of this thesis and the relationship among them. It summarizes the contributions and the impact of the thesis, as a whole research project. The thesis contributes in a significant way in the areas of Mixed Integer Optimization, Global Optimization and Structural Optimization. Briefly, articles 1 and 2 describe the theory for applying correctly the Generalized Benders' Decomposition method for Topology Optimization problems. Article 3 describes a significant improvement in the GBD method for topology design problems, that could be generalized to a large class of mixed integer problems. Article 4 shows the application of this method in the design of composite laminated structures through the discrete material optimization approach, as well as new improvements in the technique, by means of the use of heuristics. Finally, article 5 describes the inclusion of local failure criteria in the modeling, and proposed an algorithm based in the same optimization method to obtain globally optimal solutions.

GBD Method for Topology Optimization: Method and Theoretical Properties (Appendix A)

This article considers the non-linear mixed-integer optimization programs that appear in structural topology optimization. The article presents a Generalized Benders' Decomposition (GBD) method for solving single and multiple load minimum compliance (maximum stiffness) and minimum weight problems to global optimality. We present the theoretical aspects of the method when treating topology design optimization problems. We show a proof of finite convergence and conditions for obtaining global optimal solutions to single load re-enforcement problems, i.e. structural optimization problems for which no holes can be created. The method is also linked to, and compared with the Outer-Approximation approach and 0 – 1 semidefinite programming formulations of the considered problems. Article 1 contains all theoretical aspects related to the GBD method applied to topology optimization problems.

GBD Method for Topology Optimization: Extension and Numerical Experiments (Appendix B)

In Article 2, the method and the theoretical results from article 1 are generalized to pure topology optimization problems under multiple load conditions. Some heuristics, including non convex reformulation to obtain good candidate solutions, inclusion of the GBD cut related to the continuous relaxation, and the use of combinatorial Benders' feasibility cuts are suggested to accelerate the method. An implementation of the algorithm is described in detail. A set of truss topology optimization problems are solved numerically to global optimality. Article 2 is more based in numerical experiences, related to the theory built in Article 1.

New GBD Method for Topology Optimization Using Level Set Cuts (Appendix C)

This article considers the use of a new type of Generalized Benders' Decomposition (GBD) method, modifying the classical procedure. GBD solves non linear mixed integer problems, by solving a sequence of linear mixed integer problems. The classical method includes at each instance of this sequence, one or several linear constraints (or cuts), obtained from the solution of the previous problem in the sequence. The new proposed method considers the level set of the lowest available upper bound for the optimal objective function. Then, it searches for a non (necessarily) feasible design at this level set, and it forms a linear constraint (GBD cut) from this "level set" design. This new type of non feasible cuts are stronger than the classical GBD cuts and numerical results show that in practice they lead to faster convergence. The method is derived theoretically and specifically for classical structural topology problems, but it can be generalized to a larger class of non-linear mixed integer problems, where the mixed problem can be reformulated as an integer problem with a continuous convex relaxation. In this case, global solutions are guaranteed. A set of numerical benchmark examples for structural topology optimization problems are solved to global optimality. At this point the method has not yet got to its most powerful form. This is attained in article 4.

Discrete Multimaterial Optimization:Combining Approaches for Global Optimization (Appendix D)

Article 4 considers application of the theory and algorithm developed in the previous articles, but applied in the design of composite laminated structures. In addition, it includes heuristics in combination with the GBD method, so the resulting combined algorithm maximizes its computational power to convergence to global optimality. In this article we formulate composite laminate design problems as discrete material optimization problems. By using this modeling, we state standard minimum com-

pliance problems in their original Mixed Integer Problem (MIP) formulation, which we aim to solve to Global optimality. We use different techniques for continuous and discrete optimization, and a Generalized Benders' Decomposition algorithm for obtaining globally optimal solutions. By solving the continuous relaxation of the mixed integer problem, a considerable amount of information is passed to the mixed integer problem. This is mainly due to a convexity property of the continuous relaxation of the original problem. In particular, we use an efficient heuristic technique, which is very likely to find close-to optimal solutions. This technique consists in solving a corresponding sub-MIP problem, based on the solution the continuous relaxation of the original MIP optimization problem. This heuristic can be also used to improve the performance of other optimization techniques in the field of Mixed Integer Optimization. A number of numerical examples of design of composite laminated structures is presented. Several of them were solved to global optimality, and the strengths of the method are discussed in extension. Numerical examples of medium size discretization of up to 23.000 design variables gives promising results of solving large design problems to optimality, considering a small tolerance. At the same time, the independence of the design discretization with respect to the finite element discretizations allows the method to be used in real life design problems and still obtain global solutions.

Global Optmization for Structural Design Problem with Local Failure Criteria (Appendix E)

This article considers the inclusion of local failure criteria in multi-material structural design problems, stated in a non-linear mixed 0-1 formulation. Our main goal is to formulate models and methods allowing us to solve the design problem to global optimality. We use the Generalized Benders' Decomposition (GBD). The local failure criteria we consider are the maximum strain, the maximum stress, the Tsai-Hill, and the Tsai-Wu criteria. We reformulate the classical formulation of the failure criteria, into a set of convex inequalities, forming a set of convex constraints. Including these reformulations on the design problem, we obtain a mixed 0 – 1 problem with convex continuous relaxation. We can therefore use the Generalized Benders' Decomposition and/or the Outer Approximation approaches to construct an algorithm solving the design problem to global optimality. A costumized GBD algorithm able to attack minimum compliance and minimum weight problems considering any type of local failure criteria is proposed. A numerical example for fiber angle optimization is tested and solved to global optimality by use of the proposed algorithm.

2.1 Contributions and Impact

This thesis supplies contributions in several areas of research. In particular, it is the first work in Global Optimization applied to the design of composite laminated

structures which includes local failure criteria. Besides, it provides a method which can be applied for any existing local failure criteria, independently of its mathematical properties. This is due to the fact that the method only used the evaluation of the failure criterion, in an approval/rejection procedure for the candidate designs. The method is based on the algorithm related to the corresponding local failure criterion-unconstrained optimization problem. The flexibility of the method is a consequence of the fact that we do not use gradient information of the failure functions, neither we need to use any potential convexity properties of the local failure functions. In a more general perspective, this thesis produced a significant amount of theoretical and numerical contributions to the structural optimization field. In particular, it is the only existing work in applying the Generalized Benders' Decomposition method in structural design problems. In addition, all theoretical aspects regarding the convergence of the method in a finite number of steps, the convergence to global optimality of the method were studied, opening and closing this sub-research area. In addition, the thesis presents a new method for Mixed integer Optimization, which is a GBD related method. The method is developed from a theoretical point of view for structural design problems. All theoretical results from the GBD method can be obtained to the new method. The improvement in numerical performance of the proposed method is dramatic. On top of this improvement, a number of heuristics were used to improve even more the numerical performance of the method, and successful numerical examples were presented.

Article 1

Munoz, Eduardo; Stolpe, Mathias. Generalized Benders' Decomposition for Topology Optimization Problems - Part 1: Methos and Theoretical Properties. Submitted to *Journal of Global Optimization*.

Generalized Benders' Decomposition for Topology Optimization Problems – Part 1: Method and Theoretical Properties

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June 6, 2010

Abstract

This two part article considers the non-linear mixed-integer optimization programs that appear in structural topology optimization. The main objective of this work is to present a generalized Benders' decomposition (GBD) method for solving single and multiple load minimum compliance (maximum stiffness) and minimum weight problems to global optimality. The GBD technique is a classical method for mixed-integer programming, in which the non-linear mixed-integer program is replaced by a sequence of linear mixed-integer programs.

Part one presents the theoretical aspects of the method, including a proof of finite convergence and conditions for obtaining global optimal solutions to single load reinforcement problems, i.e. structural optimization problems for which no holes can be created. The method is also linked to, and compared with the Outer-Approximation approach and 0 – 1 semidefinite programming formulations of the considered problems.

In the second part, the method and the theoretical results are generalized to pure topology optimization problems under multiple load conditions. Several ways to accelerate the method are suggested and an implementation is described in detail. Finally, a set of truss topology optimization problems are numerically solved to global optimality.

Mathematical Subject Classification (2000): 90C90, 74P05, 74P15

Keywords: Structural Topology Optimization, Global Optimization, Generalized Benders' Decomposition, Outer-Approximation.

1 Introduction

We consider structural topology optimization problems with discrete design variables, in particular minimum compliance (maximum stiffness) and minimum weight problems. For an overview of the field of topology optimization we refer to [7]. We study the mentioned problems in mixed 0 – 1 formulations, from the theoretical point of view. Up to now, numerical experiences have shown that discrete formulations of topology optimization problems cannot be efficiently solved when the number of design variables becomes large. Topology optimization problems have however only recently been treated by different

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global optimization techniques, such as the branch-and-bound technique for non-linear mixed integer optimization, see e.g. [8] and [2, 3].

We propose to solve this kind of topology design problems by using generalized Benders' decomposition (GBD, see [15]). Benders developed in the 1960's a technique for solving linear mixed-integer problems (Benders' Decomposition, see [6], 1962), and it was extended to non-linear mixed-integer problems by Geoffrion (GBD, see [15], 1972) and Lazimy (GBD, extended version, [20], 1986). The proposed method solves these problems, under certain conditions, to a global optimum. This skill rarely appears in the standard non-linear optimization methods used in the field of structural topology optimization. The models and methods mainly used in the field, in general, only guarantee locally optimal designs (if they at all find feasible designs).

The GBD technique is based on solving a sequence of linear mixed integer optimization programs, which approximate the original non-linear mixed integer program. These programs can be solved by any method for linear mixed integer optimization, as for example, branch-and-cut ([24]), or branch-and-bound ([16], [19], and [27]) methods. The GBD algorithm forms a monotone sequence of lower and upper bounds of the optimum value, which we prove to converge in a finite number of iterations for the considered problems.

This article is organized as follows. Section 2 presents the mixed 0 – 1 formulations of the considered minimum compliance and minimum weight problems, as well as the assumptions considered throughout this article. Section 3 develops the theory related to the application of the GBD for the minimum compliance problem, for the re-enforcement scenario, and in the single load case. Section 4 presents the formal and precise statement of the GBD technique for the minimum compliance problem. Section 5 presents the theoretical basis to apply GBD to the single load minimum weight problem. In Section 6, the theorem of convergence of the method in a finite number of steps is stated and proved. The proof of the convergence of the algorithm to a global optimum is shown in Section 7. Section 8 presents the formulation of the Outer-Approximation method for the minimum compliance problem and its relationship with the presented GBD approach. Section 9 presents an equivalent mixed 0 – 1 semi definite programming formulation of the minimum compliance problem and its relationship with the presented GBD approach. Finally, Section 10 contains a brief summary of the results presented in this article.

In the second part of the article, the main results are generalized to pure topology optimization problems under multiple load conditions. These generalizations are important for practical applications, for example design of advanced composite structures. In the second part we also present several ways to accelerate the proposed basic GBD method. Finally, we present the details of an implementation and the numerical experience obtained from solving a set of truss topology optimization problems to global optimality.

2 Problem Statement

We consider a closed-bounded design domain $\Omega_c \subset \mathbb{R}^2$ or $\Omega_c \subset \mathbb{R}^3$, with $\partial\Omega_c$ of class \mathcal{C}^1 . The design space related to Ω_c corresponds to the power set $P(\Omega_c)$ of Ω_c (i.e., the set of all subsets of Ω_c). $P(\Omega_c)$ has infinite cardinality. For numerical purposes, the design space Ω_c is discretized into a finite set of n small design regions, $\Omega_D = \{D_1, \dots, D_n\}$, with $\cup_{i=1}^n D_i = \Omega_c$, $D_i \cap D_j = \emptyset$, if $i \neq j$. A 0 – 1 design variable is linked to each D_i . The discrete design space corresponds now to the power set of Ω_D , $P(\Omega_D)$, of cardinality 2^n . Typically (but not necessarily), these design regions coincide with the finite elements that are used to compute the response (displacements, stresses, etc.) of the structure under loading. The design variables can represent thicknesses, areas, or densities of a given isotropic design material in each of the elements. They can also, as in the case of design

of composite structures, represent a choice of available composite materials. A design is characterized by a vector element x in the discrete design set $\chi_D = \{0, 1\}^n$, where n is the number of design variables induced by the discretization. In order to ensure that the problem is mathematically well formulated, we assume that suitable boundary conditions and external loads are imposed. The design problem consists in finding the optimal vector $x^* \in \chi_D$ that minimizes the objective function (compliance, weight, etc.), under some restrictions in the design space χ_D . Throughout, we consider optimal design of mechanical structures in linear elasticity, subjected to static forces. We assume that the response of the structure is computed using the finite element method (see e.g. [9]).

The vector $x \in \chi_D$ will throughout be denoted the design variable vector, or simply the design variable. The elastic equilibrium equations, relating an external force f applied to the structure and the corresponding displacement u , are given by

$$K(x)u = f. \quad (1)$$

Here, $u \in \mathbb{R}^d$ is the vector of displacements, and $f \in \mathbb{R}^d$ represents a given external static load for each of the d degrees of freedom introduced by the discretization of the design space. The stiffness matrix $K(x) \in \mathbb{R}^{d \times d}$ is a function of the design variable x , and we assume that the load vector f is independent of the design x .

The first of the problems we study is the minimum compliance (maximum stiffness) problem, which is formulated as

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, u \in \mathbb{R}^d}{\text{minimize}} && f^T u \\ & \text{subject to} && K(x)u = f, \\ & && \rho^T x \leq M, \\ & && Ax \leq b, \\ & && x \in \chi_D = \{0, 1\}^n. \end{aligned} \quad (2)$$

In (2), $M > 0$ is the maximum amount of material allowed for the structure, and $\rho \in \mathbb{R}^n$ is the vector of material densities. The constraints $Ax \leq b$ state a general set of linear inequalities involving only the design variable. The second problem formulation we study, is the minimum weight problem

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, u \in \mathbb{R}^d}{\text{minimize}} && \rho^T x, \\ & \text{subject to} && K(x)u = f, \\ & && f^T u \leq C, \\ & && Ax \leq b, \\ & && x \in \chi_D = \{0, 1\}^n, \end{aligned} \quad (3)$$

where $C > 0$ is the maximum allowed compliance for the structure. Problems (2) and (3) are both non-linear mixed 0 – 1 programs. They are also non-convex problems, due to the integer nature of the design variables, and the non-linearity of the equilibrium equations (1).

Denote the sets

$$\begin{aligned} X_M &= \{x \in [0, 1]^n \mid \rho^T x \leq M\}, \\ X_b &= \{x \in [0, 1]^n \mid Ax \leq b\}, \\ X &= \{0, 1\}^n \cap X_M \cap X_b. \end{aligned} \quad (4)$$

2.1 Assumptions

A summary of the important assumptions we consider throughout this article is the following:

- (A-1) The stiffness matrix $K(x)$ is symmetric, affine in x , and positive definite for all $x \in \{0, 1\}^n$. The matrix $K(x)$ is given by

$$K(x) = K_0 + \sum_{j=1}^n x_j K_j, \quad (5)$$

where $K_j \in \mathbb{R}^{d \times d}$ is the symmetric positive semi definite element stiffness matrix for the j -th design variable, and $K_0 \in \mathbb{R}^{d \times d}$ is a given symmetric positive definite matrix ($K_0 \succ 0$). We assume that

$$K_0 = \eta_s \sum_{j=1}^n K_j, \quad (6)$$

with $0 < \eta_s \ll 1$. η_s will be called the re-enforcement parameter.

- (A-2) The compliance and mass limits, C and M respectively, satisfy $C > 0$ and $0 < M < \sum_{j=1}^n \rho_j$, where $\rho_j \geq 0$ for all $j = 1, \dots, n$.
- (A-3) The external load $f \in \mathbb{R}^d \setminus \{0\}$.
- (A-4) There are no special assumptions on A and b , except for the requirement that the feasible set related to the constraint $Ax \leq b$ is non empty.

The first assumption states that we are in a re-enforcement scenario. In this situation, there already exists some given design, described by the matrix K_0 , and the objective is to find out where to re-enforce this pre-existing design in an optimal way. The assumption (A-1) will be relaxed in the second part of the article, where we allow the introduction of holes in the structure. The second and third assumptions are stated to avoid trivial, or no solutions to the considered problems.

3 GBD for the Minimum Compliance Problem

In this section, we present the Generalized Benders' Decomposition (GBD), applied to the single load minimum compliance problem (2). We consider the situation of a re-enforcement scenario, where the global stiffness matrix $K(x)$ is represented by (5) and (6) in assumption (A-1), with $\eta_s > 0$. This ensures that $K(x)$ is positive definite, for all $x \geq 0$. Hence, for any design vector $x \in \{0, 1\}^n$, the equilibrium equations (1) have a unique solution. These assumptions are introduced to keep the models and results as simple as possible in this first stage. The generalization to a positive semidefinite stiffness matrix and multiple load conditions are presented on the following sections .

GBD is, roughly speaking, based on splitting the problem (2) into two sequences of simpler optimization programs. The first sequence of programs involves only the state variable u , while the other sequence considers only the 0 – 1 design variable x . The programs in the state variable u are called the “subproblems”. They correspond to solving the analysis problem (1) for a fixed value of the design x , obtaining the corresponding displacement $u(x)$. The second sequence of problems is composed of linear 0 – 1 problems

in the design variable x . Each one of these linear 0 – 1 programs is called a “relaxed master problem”, and their output decides the convergence of the entire algorithm.

It is possible to prove, by using the projection theorem (see [12], Section 2.1, for details), that the minimum compliance problem (2) is equivalent to the projected problem

$$\begin{aligned} & \underset{x \in \mathbb{R}^n}{\text{minimize}} && w(x) \\ & \text{subject to} && \rho^T x \leq M, \\ & && Ax \leq b, \\ & && x \in \{0, 1\}^n, \end{aligned} \tag{7}$$

where $w(x)$ is the solution to the analysis problem

$$\begin{aligned} w(x) = & \underset{u \in \mathbb{R}^d}{\text{minimize}} && f^T u \\ & \text{subject to} && K(x)u = f. \end{aligned} \tag{8}$$

The problem (2) satisfies all the hypotheses stated in the projection theorem, and so, it implies the equivalence between the problems (2) and the problem defined by (7) - (8). This equivalence should be understood in the sense that both programs have the same optimal solutions and objective values. Since we have split problem (2) into two eventually simpler problems (7) - (8), our goal is to solve these two last problems, instead of the problem (2). The analysis problem (8) is a linear problem, which in the re-enforcement case (i.e., $\eta_s > 0$ in assumption (A-1)) has a unique solution. Thus, we can find an explicit representation for the function $w(x)$, since the optimal solution of the program (8) is the vector $u^* = u(x) = K(x)^{-1}f$. In this case, we have that $w(x)$ is given by

$$w(x) = f^T u(x) = f^T K(x)^{-1}f. \tag{9}$$

The original optimization problem given by (7) - (8) can, in this situation, be reduced to the program

$$\begin{aligned} & \underset{x \in \mathbb{R}^n}{\text{minimize}} && f^T K(x)^{-1}f \\ & \text{subject to} && \rho^T x \leq M, \\ & && Ax \leq b, \\ & && x \in \{0, 1\}^n, \end{aligned} \tag{10}$$

which is a program stated only in the design variable x . Problem (10) is the classical nested formulation of a minimum compliance problem (see e.g. [7] and [25]). It is a non-linear 0 – 1 program, and its continuous relaxation (i.e., the problem obtained when the condition $x \in \{0, 1\}^n$ is replaced by $x \in [0, 1]^n$) is convex, as [25] showed that the function $w(x) = f^T K(x)^{-1}f$ is convex for $x \in [0, 1]^n$, such that $x > 0$. Here, because of assumption (A-1), convexity of $w(x)$ holds in the whole space $[0, 1]^n$.

The original idea of the GBD (see [15] for details), as a solution strategy, is to use the dual formulation related to the subproblem (8), to define the function $w(x)$. It is expected that by using the dual formulation, a linear inequality involving the design variable x will appear (these linear inequalities are denoted cuts). Note, that since the program (8) is a linear program, it follows that it satisfies the strong duality property, i.e. the primal and dual optimization programs have the same optimal value if they are both feasible. This last fact allows us to write the function $w(x)$, using the Lagrange function of the program (8), as

$$w(x) = \supremum_{\alpha \in \mathbb{R}^d} \{ \infimum_{u \in \mathbb{R}^d} \{ f^T u + \alpha^T [K(x)u - f] \} \}. \tag{11}$$

By using linear duality theory, (11) can be re-written as the linear problem

$$\begin{aligned} w(x) = \quad & \underset{\alpha \in \mathbb{R}^d}{\text{maximize}} && -f^T \alpha \\ & \text{subject to} && K(x)\alpha = -f, \end{aligned}$$

which is similar to the analysis problem (8). The optimal solution of this last program is given by $\alpha^* = -u^* = -K(x)^{-1}f$. The couple (α^*, u^*) is then, the optimal solution of the primal-dual problem (11). Replacing it in (11), we obtain that $w(x)$ can be represented by the function

$$\begin{aligned} w(x) = w(\alpha^*, u^*, x) &= 2f^T K(x)^{-1}f - f^T (K(x)^{-1})^T K(x) K(x)^{-1}f \\ &= f^T K(x)^{-1}f, \end{aligned}$$

and we obtain the same representation of the function $w(x)$ as in (9). Thus, no improvements were made with this formulation, and neither did any linear relationship with respect to x appear. As an alternative, [20] proposed to modify the primal formulation of $w(x)$, in such a way that the dual definition of $w(x)$, defines by construction a set of linear inequalities in the design variable x . These linear inequalities can be used to form an equivalent linear optimization program, which is useful for numerical purposes. In order to have such a representation of the function $w(x)$, by using dual information, we need to satisfy the strong duality property. The idea of [20], applied to the function $w(x)$ defined by (8), is given by the following non convex subproblem,

$$\begin{aligned} w(x) = \quad & \underset{z \in \mathbb{R}^n, u \in \mathbb{R}^d}{\text{infimum}} && f^T u \\ & \text{subject to} && K(z)u = f, \\ & && z = x. \end{aligned} \tag{12}$$

Hence, the corresponding dual definition of the function $w(x)$, in case the strong duality property held, is

$$\begin{aligned} w(x) &= \underset{\alpha \in \mathbb{R}^d, \psi \in \mathbb{R}^n}{\text{supremum}} \{ \underset{z \in \mathbb{R}^n, u \in \mathbb{R}^d}{\text{infimum}} \{ f^T u + \alpha^T [K(z)u - f] + \psi^T (z - x) \} \} \\ &= \underset{\alpha \in \mathbb{R}^d, \psi \in \mathbb{R}^n}{\text{supremum}} \{ -\psi^T x - f^T \alpha + \underset{z \in \mathbb{R}^n, u \in \mathbb{R}^d}{\text{infimum}} \{ f^T u + \alpha^T K(z)u + \psi^T z \} \}. \end{aligned} \tag{13}$$

Unfortunately, this property does not hold for the problem (12). This can be seen by looking in (13) at the program

$$P(\alpha, \psi) = \underset{z \in \mathbb{R}^n, u \in \mathbb{R}^d}{\text{infimum}} \{ f^T u + \alpha^T K(z)u + \psi^T z \},$$

and the following result.

Proposition 1. $P(\alpha, \psi) = -\infty$, for all $(\alpha, \psi) \in \mathbb{R}^d \times \mathbb{R}^n$.

Proof. The affine representation of the stiffness matrix (5), is represented in a more compact way as $K(z) = K_0 + (\nabla_z K(z))z$. Here,

$$\frac{\partial K(z)}{\partial z_j} = K_j,$$

where K_j is the local stiffness matrix related to the j -th design variable. It follows that the gradient $\nabla_z K(z)$ of the stiffness matrix $K(z)$ is expressed by the “vector” of local stiffness matrices

$$\nabla_z K = \nabla_z K(z) = \{K_1 \ K_2 \ \dots \ K_n\}. \quad (14)$$

Therefore, we have

$$\begin{aligned} P(\alpha, \psi) &= \inf_{z \in \mathbb{R}^n, u \in \mathbb{R}^d} \{f^T u + \alpha^T [K_0 + (\nabla_z K)z]u + \psi^T z\} \\ &= \inf_{z \in \mathbb{R}^n, u \in \mathbb{R}^d} \{[f^T + \alpha^T K_0]u + \psi^T z + \alpha^T (\nabla_z K)zu\}. \end{aligned}$$

Here we see that $P(\alpha, \psi)$ represents an unconstrained minimization problem, which is bilinear in the variables $z \in \mathbb{R}^n$ and $u \in \mathbb{R}^d$. The coefficients involved in this problem never become simultaneously zero, for any value of the Lagrange multipliers. In other words, the equations

$$(f^T + \alpha^T K_0, \psi^T, \alpha^T \nabla_z K) = (0, 0, 0), \quad \forall (\alpha, \psi) \in \mathbb{R}^d \times \mathbb{R}^n, \quad (15)$$

have no solutions. This can be seen in the following way. If $\eta_s = 0$, then $K_0 = 0^{d \times d}$, and the vector at the left side in (15) becomes $(f^T, \psi^T, \alpha^T \nabla_z K)$, which can not become zero, since its first component $f^T \neq 0$ is independent of (α, ψ) . If now $\eta_s > 0$, Then K_0 is a positive definite matrix. Suppose there exists a couple (α^*, ψ^*) satisfying (15). Then α^* must be the unique solution of the linear system $f + \alpha^{*T} K_0 = 0$ (the existence and uniqueness of α^* come from the positive definiteness of the symmetric matrix K_0). Therefore, $\alpha^* \neq 0$ (otherwise it would imply $f = 0$, contradicting assumption (A-3)). The third system of equations, $\alpha^T \nabla_z K = 0$ implies that

$$K_j \alpha^* = 0, \quad \forall j = 1, \dots, n. \quad (16)$$

Now, the system $f + \alpha^{*T} K_0 = 0$ and the definition of K_0 given by (6), necessarily mean that

$$\begin{aligned} f + \alpha^{*T} \eta_s \sum_{j=1}^n K_j &= 0 \\ \iff f + \eta_s \sum_{j=1}^n \alpha^{*T} K_j &= 0, \end{aligned} \quad (17)$$

and after (16), it follows that

$$f = 0, \quad (18)$$

which is a contradiction with assumption (A-3). This proves the insolvability of (15). As a consequence, the minimization problem defined by $P(\alpha, \psi)$ is unbounded, independently of the values of (α, ψ) , i.e.,

$$P(\alpha, \psi) = -\infty, \quad \forall (\alpha, \psi) \in \mathbb{R}^d \times \mathbb{R}^n.$$

□

This last result implies that for the primal subproblem (12), no optimal Lagrange multipliers are available. Therefore, the representation of $w(x)$ given by (13) is not equivalent to (9), and hence, it is not valid. This problem arises from the lack of a necessary hypothesis called stability (see [12]) for the primal subproblem, which [20] assumes when formulating the method. This hypothesis is not satisfied here, and the non existence of the optimal Lagrange multipliers is itself, a proof of it. The solution to this difficulty is to replace once more the primal definition of the function $w(x)$, for a new equivalent one, where these two properties (duality and linearity of dual in the sense explained previously) hold at the same time.

3.1 GBD and Duality-Linearity Properties

In the GBD method, we must replace the definition of the function $w(x)$ by an equivalent one, where the strong duality property of its formulation holds. It is important as well, that $w(x)$ defines a set of linear inequalities with respect to the design variable x , as it was the original idea of [20]. With this purpose in mind, we propose to define the function $w(x)$ as

$$w(x) = \tilde{w}(x, u(x)), \quad (19)$$

where $u(x) = K(x)^{-1}f$ and $\tilde{w} : \{0, 1\}^n \times \mathbb{R}^d \rightarrow]0, \infty]$ is the function defined by the program

$$\begin{aligned} \tilde{w}(x, u) = \quad & \inf_{z \in \mathbb{R}^n, v \in \mathbb{R}^d} f^T v \\ \text{subject to} \quad & K(z)v - f = 0, \\ & v = u, \\ & vz^T = ux^T, \end{aligned} \quad (20)$$

when this program is feasible, and $\tilde{w}(x, u) = +\infty$, when it is not. Here, the use of the semi open set $]0, \infty]$ for the image of $\tilde{w}(x, u)$ follows the fact that the function \tilde{w} can not attain the value 0 (If $\tilde{w}(x, u)$ attained 0, from the positive definiteness of $K(x)$, it would imply $v^* = u = 0^T$ in (20), which can never satisfy the constraint $K(z)v - f = 0$). Note the special notation in the last constraint of (20). $vz^T = ux^T$ is a simplification of the notation for the representation of the set of constraints: $v_j z_k = u_j x_k$, for $j \in \{1, \dots, d\}$ and $k \in \{1, \dots, n\}$. Keeping this in mind, we will from now on simply write $vz = ux$.

After introducing this new definition for $w(x)$, we need to show that this definition is equivalent to (9). This means, we need to prove that

$$w(x) = \tilde{w}(x, u(x)), \quad \forall x \in \{0, 1\}^n.$$

We start by taking an arbitrary $x \in \{0, 1\}^n$. It is possible to see that the program (20) only has feasible points of the type (x, u) satisfying $u = u(x)$, where $u(x) = K(x)^{-1}f$. For such couples $(x, u(x))$, the only feasible, and therefore optimal solution to (20), corresponds to

$$\begin{aligned} z^* &= x, \\ v^* &= u(x), \end{aligned}$$

in which case we have

$$\tilde{w}(x, u(x)) = f^T u(x) = w(x). \quad (21)$$

Since (21) is valid for an arbitrary $x \in \{0, 1\}^n$, this proves that the definition (19) for the function $w(x)$ is equivalent to (9). Next, we prove that the strong duality property holds for the program (20). The dual optimization problem related to (20) can be written as

$$\Phi_c(x, u) = \sup_{\Gamma_c \in \mathbb{R}^{2d+nd}} \{l_c^*(x, u, \Gamma_c)\}, \quad (22)$$

where the following definitions are introduced. Γ_c is defined as the triple

$$\Gamma_c = (\alpha, \delta, \nu), \quad \alpha \in \mathbb{R}^d, \delta \in \mathbb{R}^d, \nu \in \mathbb{R}^{nd}.$$

The function $l_c^* : \{0, 1\}^n \times \mathbb{R}^d \times \mathbb{R}^{2d+nd}$ is defined as

$$\begin{aligned} l_c^*(x, u, \Gamma_c) &:= \inf_{z \in \mathbb{R}^n, v \in \mathbb{R}^d} \{\mathcal{L}^*(x, u, \Gamma_c, z, v)\}, \\ \mathcal{L}^*(x, u, \Gamma_c, z, v) &:= f^T v + \alpha^T [f - K(z)v] + \delta^T [v - u] + \nu^T [vz - ux]. \end{aligned} \quad (23)$$

In (23), the natural way to define the dual variable ν , related to the third constraint in (20), is by using two artificial indexes l, k , but we remark that ν is a 1-dimensional vector and this notation is not ambiguous

$$\begin{aligned}\nu \in \mathbb{R}^{nd} : \quad \nu^T[vz - ux] &= \sum_{l,k} \nu_{lk}[v_l z_k - u_l x_k] \\ &= \sum_{l,k} \nu_{lk}[z_k v_l - x_k u_l] \\ &= \nu^T(zv - xu).\end{aligned}\tag{24}$$

The first important result we can show in this section is the following strong duality result.

Theorem 1. *The program (20) is a non-convex program. Even so the following strong duality property holds*

$$\tilde{w}(x, u(x)) = \Phi_c(x, u(x)) = f^T u(x).\tag{25}$$

Proof. The program (20) is non convex, since its first and third constraints are bilinear constraints in the $v - z$ plane. To prove that the strong duality property (25) holds, we need to prove that both the primal and dual programs (20) and (22), respectively, are feasible for the same values of (x, u) , and their optimal objective function values are equal.

Consider $u = u(x) = K(x)^{-1}f$. Since we already know from (21) that $\tilde{w}(x, u(x)) = f^T u(x)$, then we only need to prove that the dual program (22) has the same optimal objective function value, i.e. that

$$\Phi_c(x, u(x)) = f^T u(x).$$

We remind the reader that the weak duality theorem holds always, even for non convex or non regular optimization programs (see [17], p. 149 or [14], p. 9), without any additional hypotheses. This means that we have the following property

$$\Phi_c(x, u(x)) \leq \tilde{w}(x, u(x)) = f^T u(x).\tag{26}$$

We need to prove that in (26), the inequality is actually, an equality. We start by looking at the feasible set of the dual program (22). The weak duality property claims, more specifically, that for minimization problems, the objective value of any feasible solution of the dual problem, is not larger than the objective value of any feasible solution for the primal problem. In particular, it is not larger than the optimal value of the primal problem

$$l_c^*(x, u(x), \Gamma_c) \leq f^T u(x), \quad \forall \Gamma_c, \quad \text{dual feasible.}$$

By definition, the feasible set $F_D(x, u)$ of the dual program (22) is the subset of the space of the Lagrange multipliers, for which the value of the dual objective function is finite

$$F_D(x, u) = \{\Gamma_c \in \mathbb{R}^{2d+nd} : l_c^*(x, u, \Gamma_c) > -\infty\}.$$

If the strong duality (25) property holds, it necessarily follows that for any $x \in \{0, 1\}^n$, we are able to find an optimal Lagrange multiplier, $\Gamma_c(x)$, such that

$$l_c^*(x, u(x), \Gamma_c(x)) = f^T u(x).$$

It will be proven that the equality is attained for the following values of the multiplier vectors

$$\begin{aligned}
\Gamma_c(x) &= (\alpha(x), \delta(x), \nu(x)) \\
\alpha(x) &= u(x) \\
\delta(x) &= -x^T \nabla_x K(x) u(x) \\
\nu(x) &= \nabla_x K(x) u(x),
\end{aligned} \tag{27}$$

where $\nabla_x K(x)$ is the gradient of the stiffness matrix given by (14), for the variable x instead of z . We perform the replacements by steps. This will make the algebraic manipulation more clear. As a first step, we start by replacing $\alpha(x) = u(x)$ in (23). By doing this, (23) becomes

$$\begin{aligned}
l_c^*(x, u(x), \alpha(x), \delta, \nu) &= \inf_{z \in \mathbb{R}^n, v \in \mathbb{R}^d} \{ \mathcal{L}^*(x, u(x), \alpha(x), \delta, \nu, z, v) \} \\
&= \inf_{z \in \mathbb{R}^n, v \in \mathbb{R}^d} \{ f^T v + u(x)^T [f - K(z)v] + \delta^T [v - u(x)] \\
&\quad + \nu^T [vz - u(x)x] \} \\
&= \inf_{z \in \mathbb{R}^n, v \in \mathbb{R}^d} \{ [f^T - u(x)^T K(z)]v + \delta^T [v - u(x)] \\
&\quad + \nu^T [vz - u(x)x] \} + f^T u(x).
\end{aligned}$$

Now we make the changes of variables $z = x + \Delta z$, $v = u(x) + \Delta v$, and use that the stiffness matrix $K(z)$ is affine, so we have $K(x + \Delta z) = K(x) + \nabla_x K(x) \Delta z$. Then, we obtain

$$\begin{aligned}
l_c^*(x, u(x), \alpha(x), \delta, \nu) &= \inf_{\Delta z \in \mathbb{R}^n, \Delta v \in \mathbb{R}^d} \{ [f^T - u(x)^T K(x + \Delta z)] [u(x) + \Delta v] + \delta^T \Delta v \\
&\quad + \nu^T [[u(x) + \Delta v][x + \Delta z] - u(x)x] \} + f^T u(x) \\
&= \inf_{\Delta z \in \mathbb{R}^n, \Delta v \in \mathbb{R}^d} \{ [f^T - u(x)^T [K(x) + \nabla_x K(x) \Delta z]] [u(x) + \Delta v] \\
&\quad + \delta^T \Delta v + \nu^T [u(x) \Delta z + x \Delta v + \Delta v \Delta z] \} + f^T u(x) \\
&= \inf_{\Delta z \in \mathbb{R}^n, \Delta v \in \mathbb{R}^d} \{ -u(x)^T \nabla_x K(x) \Delta z [u(x) + \Delta v] + \delta^T \Delta v \\
&\quad + \nu^T [u(x) \Delta z + x \Delta v + \Delta v \Delta z] \} + f^T u(x),
\end{aligned}$$

where in the last equation above we used the equality $f - K(x)u(x) = 0$ in order to cancel some of the terms. After some algebraic manipulations, we obtain

$$\begin{aligned}
l_c^*(x, u(x), \alpha(x), \delta, \nu) &= \inf_{\Delta z \in \mathbb{R}^n, \Delta v \in \mathbb{R}^d} \{ [\nu^T - u(x)^T \nabla_x K(x)] [\Delta z u(x) + \Delta z \Delta v] \\
&\quad + [\delta^T + \nu^T x] \Delta v \} + f^T u(x).
\end{aligned}$$

At this point, we can see that if we choose the vectors $\nu(x)^T = u(x)^T \nabla_x K(x)$ and $\delta(x)^T = -\nu(x)^T x$, we obtain

$$l_c^*(x, u(x), \Gamma_c(x)) = f^T u(x). \tag{28}$$

Equation (28), together with the weak duality property (26) and definition (20), prove that $\Gamma_c(x) = (\alpha(x), \delta(x), \nu(x))$ chosen as (27), is an optimal solution for the dual problem $\Phi_c(x, u(x))$, with an optimal value

$$\Phi_c(x, u(x)) = f^T u(x),$$

which means that strong duality property holds, as it was stated. \square

Corollary 1. *The following equalities hold*

$$w(x) = \sup_{\Gamma_c} \{l_c^*(x, u(x), \Gamma_c)\} = l_c^*(x, u(x), \Gamma_c(x)) = f^T u(x).$$

In addition, we have the following property for the l_c^* -functions.

Property 1. *The function l_c^* defined by (23) is linear in x , and it has an explicit representation, when the arguments are the displacement field $u^k = u(x^k)$, and any optimal Lagrange multipliers Γ_c^k (for example, the ones given by (27)) related to a design x^k . In this case, the function l_c^* is represented as*

$$l_c^*(x, u^k, \Gamma_c) = \begin{cases} l_c^*(x, u^k, \nu^k) = f^T u^k + \nu^{kT} u^k [x^k - x], & \text{if } u = u^k, \Gamma_c = \Gamma_c^k, \\ -\infty, & \text{otherwise.} \end{cases} \quad (29)$$

Proof. Representation (29) is obtained by an algebraic manipulation of the definition for the l_c^* -function (23).

$$\begin{aligned} l_c^*(x, u^k, \Gamma_c^k) &= \inf_{z \in \mathbb{R}^n, v \in \mathbb{R}^d} \{f^T v + \alpha^{kT} [f - K(z)v] + \delta^{kT} [v - u^k] + \nu^{kT} [vz - u^k x]\} \\ &= \nu^{kT} u^k [x^k - x] \\ &\quad + \inf_{z \in \mathbb{R}^n, v \in \mathbb{R}^d} \{f^T v + \alpha^{kT} [f - K(z)v] + \delta^{kT} [v - u^k] + \nu^{kT} [vz - u^k x^k]\} \\ &= \nu^{kT} u^k [x^k - x] + l_c^*(x^k, u^k, \Gamma_c^k) \\ &= \nu^{kT} u^k [x^k - x] + f^T u^k \\ &= l_c^*(x, u^k, \nu^k). \end{aligned}$$

For any $\Gamma_c \neq \Gamma_c^k$, $l_c^*(x, u^k, \Gamma_c)$ represents an unbounded minimization problem, which implies that $l_c^*(x, u, \Gamma_c) = -\infty$. \square

Note the simplified notation $l_c^*(x, u^k, \nu^k)$ that has been introduced here. According to (29), this notation is more convenient, and will be used instead of $l_c^*(x, u^k, \Gamma_c^k)$. The Lagrange multiplier ν^k can be expressed (after (27)) as

$$\nu^k = \begin{pmatrix} u^{kT} K_1 & u^{kT} K_2 & \dots & u^{kT} K_n \end{pmatrix}^T. \quad (30)$$

As a consequence, the function $l_c^*(x, u^k, \nu^k)$ can be explicitly represented, when its arguments are the Lagrange multipliers given by (27)

$$\begin{aligned} l_c^*(x, u^k, \nu^k) &= f^T u^k + \nu^{kT} u^k x^k - \nu^{kT} u^k x \\ &= f^T u^k + \sum_{j=1}^n x_j^k u^{kT} K_j u^k - \sum_{j=1}^n x_j u^{kT} K_j u^k. \end{aligned} \quad (31)$$

3.2 Master Problem

The results from the previous section allow us to give a sequence of equivalent formulations of the minimum compliance program (2), leading to the proposed GBD formulation of this problem. We start with the following result.

Proposition 2. *Suppose $(x^1, u^1), \dots, (x^P, u^P)$, are all feasible points for problem (2). Suppose as well that ν^1, \dots, ν^P are the corresponding optimal Lagrange multiplier vectors*

given by (27). If (x^*, u^*) is an optimal solution for problem (2), then (x^*, y^*) , $y^* = f^T u^*$ is an optimal solution for the problem

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\ & \text{subject to} && l_c^*(x, u^k, \nu^k) \leq y, \quad \forall k = 1, \dots, P, \\ & && \rho^T x \leq M, \\ & && Ax \leq b, \\ & && x \in \{0, 1\}^n. \end{aligned} \tag{32}$$

Proof. After the optimality of (x^*, u^*) for problem (2), it follows that (x^*, u^*) satisfies the conditions $\rho^T x^* \leq M$, $Ax^* \leq b$, $x^* \in \{0, 1\}^n$, and $K(x^*)u^* = f$. Considering an arbitrary k in $\{1, \dots, P\}$, and evaluating the l_c^* -function (given by (29)) in x^* , we have

$$l_c^*(x^*, u^k, \nu^k) = f^T u^k + \nu^{kT} u^k [x^k - x^*]. \tag{33}$$

For ν^k given by (27), (33) can be written as

$$l_c^*(x^*, u^k, \nu^k) = f^T u^k + u^{kT} \nabla_x K(x) u^k [x^k - x^*] = f^T u^k + \sum_{j=1}^n u^{kT} K_j u^k [x_j^k - x_j^*].$$

After (9), $w(x) = f^T K(x)^{-1} f$ is the explicit representation of the compliance as a function of x . It is known that $-\sum_{j=1}^n x_j u(x')^T K_j u(x') = \nabla_x w(x')^T x$ for an arbitrary $x' \in \{0, 1\}^n$ (see [7]), and that $w(x^k) = f^T K(x^k)^{-1} f = f^T u^k$. Hence, we can write

$$l_c^*(x^*, u^k, \nu^k) = w(x^k) + \nabla_x w(x^k)^T [x^* - x^k]. \tag{34}$$

Since the compliance $w(x)$ is a convex function in $[0, 1]^n$ (see [25]), we have that

$$w(x^k) + \nabla_x w(x^k)^T [x^* - x^k] \leq w(x^*) = y^*. \tag{35}$$

Equations (34) and (35) imply that

$$l_c^*(x^*, u^k, \nu^k) \leq y^*. \tag{36}$$

Inequality (36) was obtained for an arbitray $k \in \{1, \dots, P\}$. Therefore, this inequality is satisfied for all $k = 1, \dots, P$.

$$l_c^*(x^*, u^k, \nu^k) \leq y^*, \quad \forall k = 1, \dots, P,$$

which is enough to prove that (x^*, y^*) is feasible for (32). Now, suppose that $(x^*, y^* = w(x^*) = f^T u^*)$ is not optimal for (32). Thus, there exists (x^{**}, y^{**}) , optimal solution of (32) with $y^{**} < y^*$. The existence of (x^{**}, y^{**}) follows from the feasibility of (32), and from the fact that the linear functions

$$h^k : [0, 1]^n \longrightarrow l_c^*(x, u^k, \nu^k), \quad k = 1, \dots, P,$$

are bounded from below. In addition, we have that $x^{**} \neq x^*$, since x^* does not satisfy the constraint $l_c^*(x^*, u^*, \nu^*) \leq y^{**}$, since $l_c^*(x^*, u^*, \nu^*) = w(x^*) = y^* > y^{**}$. On the other hand, there exist the corresponding vectors u^{**} and ν^{**} . Then, $l_c^*(x^{**}, u^{**}, \nu^{**}) = w(x^{**}) \leq y^{**}$ implies that $w(x^{**}) < w(x^*)$. It is possible to check that $(x^{**}, u(x^{**}))$ is a feasible point for problem (2), with a smaller objective function than (x^*, u^*) . This is a contradiction with the optimality of (x^*, u^*) for problem (2), proving the optimality of (x^*, y^*) for the program (32). \square

An important consequence of Proposition 2 is that the cuts

$$l_c^*(x, u^k, \nu^k) \leq y, \quad k \in \{1, \dots, P\},$$

are valid cuts, being always satisfied for any optimal solution (x^*, u^*) of problem (2).

3.3 GBD Approach

Program (32) is called the master problem, and includes a too large number of constraints (P grows exponentially with the size of the problem). Obviously, this problem is impossible to solve as stated in practice. The solution to this difficulty is to relax the master problem (32), by considering only a few of the l_c^* -constraints. Thus, after including $N \ll P$ of these constraints $l_c^*(x, u^k, \nu^k) \leq y$, $k = 1, \dots, N$, the relaxed master problem is set to

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\ & \text{subject to} && l_c^*(x, u^k, \nu^k) \leq y, \quad \forall k = 1, \dots, N, \\ & && \rho^T x \leq M, \\ & && Ax \leq b, \\ & && x \in \{0, 1\}^n. \end{aligned} \tag{37}$$

The idea behind GBD is that for a certain $N \ll P$, the solutions of the programs (32) and (37) are the same.

In this way, each l_c^* function represents a linear constraint in the master problem, which is called an *optimality* cut, or a *compliance* cut. Thus, the optimality cut related to x^k is given by

$$-\nu^k T u^k x - y \leq -f^T u^k - \nu^k T u^k x^k.$$

The relaxed master problem for the minimum compliance problem can, thus, be represented explicitly as the linear mixed 0 – 1 program

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\ & \text{subject to} && -\nu^k T u^k x - y \leq -f^T u^k - \nu^k T u^k x^k, \quad k = 1, \dots, N, \\ & && \rho^T x \leq M, \\ & && Ax \leq b, \\ & && x \in \{0, 1\}^n, \end{aligned} \tag{P_c}$$

where ν^k is given by (30). (P_c) is the representation of the relaxed master program used in the numerical experiments in the second part of the article.

3.4 Principle Behind the GBD Algorithm

The principle behind the GBD algorithm is the following. At some stage of the algorithm (say at iteration N), N relaxed master problems have been solved. Let us suppose for simplicity, that for each relaxed master problem, exactly one optimal solution has been found. The situation can be easily generalized if the relaxed master problem possesses more than one solution. Therefore, there are in total N candidate designs. For each of them, the compliance $w^k = w(x^k) = f^T u(x^k)$, $k = 1, \dots, N$, has been computed, and the smallest of them is set as the current upper bound UB^N for the minimum compliance design, i.e.,

$$UB^N = \min\{w^1, w^2, \dots, w^N\}.$$

If w^* is the global optimal compliance of the original, i.e. the non relaxed, master program (32), then we have that

$$w^* = \min_k \{w^k\} \leq UB^N. \tag{38}$$

The existence of w^* is guaranteed from the finiteness of $\{0, 1\}^n$. Let Y^N be the optimal value of the relaxed master problem (P_c) at iteration N . Then, consider the fact that this master problem corresponds to the relaxed master problem of the previous iteration $N - 1$, plus one additional constraint. This extra constraint is the one added at the current iteration. This implies that the relaxed master problem at iteration N is necessarily more constrained than the relaxed master problem at iteration $N - 1$. Hence, there is consequently a relationship between the optimal values of two consecutive relaxed master problems

$$Y^{N-1} \leq Y^N.$$

It follows that the sequence of optimal values of the relaxed master problems solved until iteration N , forms a monotone non decreasing sequence

$$Y^1 \leq Y^2 \leq \dots \leq Y^{N-1} \leq Y^N.$$

Remember that the relaxed master problem is a simplified version of the original master problem, where most of the constraints have been removed. We conclude that the solution of the relaxed master problem (P_c), is bounded at any iteration by the solution of the original master problem (32), i.e.,

$$Y^N \leq w^* \tag{39}$$

The inequalities (38) and (39) can be written together as

$$Y^N \leq w^* \leq UB^N.$$

The GBD technique is based on the expectation that these two sequences of bounds (Y^N, UB^N) will converge to the optimal value w^* in a finite number of iterations. This assumption is indeed, proven in the next sections. For a more detailed description of the algorithm in a general framework, we refer the reader to [15].

4 Statement of the Method

In this section, we present the formal statement of the Generalized Benders' Decomposition (GBD) method to solve the minimum compliance problem (2). The assumptions considered are the same as in the previous sections, i.e. assumptions (A-1) – (A-4). The stiffness matrix is given by assumption (A-1) and equations (5) and (6), with re-enforcement parameter $\eta_s > 0$. Generalizations to the pure topology case and the multiple load case problems are shown in the second part of this article [22].

Algorithm 1

1. Set $N = 1$, the upper bound $UB = +\infty$, the lower bound $y^* = -\infty$ and the convergence tolerance $\epsilon \geq 0$. Find an initial design $x^1 \in [0, 1]^n$ (x^1 does not have to satisfy $x^1 \in \{0, 1\}^n$) satisfying $\rho^T x^1 \leq M$ and $Ax^1 \leq b$. Compute the displacement field $u^1 = K(x^1)^{-1}f$ and the compliance $C^1 = f^T u^1$.
2. Solve the first relaxed master problem

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\ & \text{subject to} && - \sum_{j=1}^n x_j u^{1T} K_j u^1 - y \leq -f^T u^1 - \sum_{j=1}^n x_j^1 u^{1T} K_j u^1, \\ & && \rho^T x \leq M, \\ & && Ax \leq b, \\ & && x \in \{0, 1\}^n, \end{aligned} \tag{M1}$$

by any solver for linear-mixed integer programming. If the problem is infeasible, stop and exit. Increase the mass bound M and restart. If the problem is feasible, it necessarily has at least one optimal solution. This comes from the fact that the function

$$h : \{0, 1\}^n \longrightarrow \mathbb{R}, \quad h(x) = u^{1T} K_j u^1 \mathbf{1}^T (x^1 - x)$$

is bounded from below. Consequently, the value of y in the program (M1) is bounded from below. Denote the found solution of (M1) by (x^*, y^*) , and its optimal value y^* . Set $N = 2$ and set the solution index $i^* = 2$.

3. Set $x^N = x^*$. Find a displacement field u^N satisfying the equilibrium equations and compute its compliance $C^N = f^T u^N$. If $C^N < UB$, then set $UB = C^N$ and $i^* = N$.
4. If $UB - y^* \leq \epsilon$, then stop. The optimal design found is x^{i^*} , with optimal value $f^T u^{i^*}$. Otherwise, go to step 5.
5. Solve the relaxed master problem

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\ & \text{subject to} && -\nu^k u^k x - y \leq -f^T u^k - \nu^k u^k x^k, \quad k = 1, \dots, N, \\ & && \rho^T x \leq M, \\ & && Ax \leq b, \\ & && x \in \{0, 1\}^n, \end{aligned} \tag{40}$$

with

$$\nu^k = ((K_1 u^k)^T \quad (K_2 u^k)^T \quad \dots \quad (K_n u^k)^T)^T,$$

by any solver for linear-mixed 0 – 1 programming. Denote the solution of this program (x^*, y^*) , and its optimal value y^* . Set $N \leftarrow N + 1$. Return to step 3.

5 GBD for Minimum Weight Problems

In this section, we discuss the details of a Generalized Benders' decomposition (GBD) method for the minimum weight problem (3), under assumptions (A-1) – (A-4). We follow the same steps as for the minimum compliance problem (2), developed in Section 3. We start by remarking that the weight function is a linear function in the design variables, and it is independent of the displacements. Therefore, only one optimality cut is necessary to represent the weight function exactly. This cut is identical to the weight function

$$l_w^*(x) = \rho^T x.$$

5.1 Feasibility of the Compliance

The next aspect to consider is the situation where the current design at a certain iteration of the algorithm does not satisfy the compliance constraint $f^T u \leq C$. This case is treated by including a set of feasibility cuts for the compliance constraint in the master problem. In order to describe these compliance feasibility cuts, the following result will be useful.

Proposition 3. *Let x be a given design for which the equilibrium equations $K(x)u - f = 0$ possess at least one solution $u(x)$. Then its compliance is infeasible, i.e., $f^T u(x) > C$, if and only if $\forall \alpha \in \mathbb{R}^d, \xi \geq 0, \delta \in \mathbb{R}^d, \zeta \in \mathbb{R}^{nd}, \psi \in \mathbb{R}^n$, it holds that*

$$l_*^c(x, u(x), \alpha, \xi, \delta, \zeta, \psi) < 0, \tag{41}$$

where l_*^c is defined as

$$l_*^c(x, u, \alpha, \xi, \delta, \zeta, \psi) := \sup_{z \in \mathbb{R}^n, v \in \mathbb{R}^d} \{ \alpha^T [K(z)v - f] + \xi [C - f^T v] + \delta^T [v - u] + \zeta^T [zv - xu] + \psi^T [z - x] \}. \quad (42)$$

Proof. Let us introduce, for simplicity, the following notation

$$l_*^c(x, u, \Gamma_w) := l_*^c(x, u, \alpha, \xi, \delta, \zeta, \psi), \quad (43)$$

where $\Gamma_w \in \mathbb{R}^{2d+n+nd} \times \mathbb{R}_+$ is defined as the vector

$$\Gamma_w = (\alpha, \xi, \delta, \zeta, \psi), \quad \alpha \in \mathbb{R}^d, \xi \in \mathbb{R}_+, \delta \in \mathbb{R}^d, \zeta \in \mathbb{R}^{nd}, \psi \in \mathbb{R}^n.$$

Consider first an arbitrary x , such that the equilibrium $K(x)u - f = 0$ possesses at least one solution $u(x)$. Suppose that its compliance satisfies $f^T u(x) \leq C$. Then, evaluating $z = x$, $v = u(x)$ in (42) and (43), we obtain the inequality

$$\begin{aligned} l_*^c(x, u(x), \Gamma_w) &\geq \alpha^T [K(x)u(x) - f] + \xi [C - f^T u(x)] + \delta^T [u(x) - u(x)] \\ &\quad + \zeta^T [xu(x) - xu(x)] + \psi^T [x - x] \\ &\geq \xi (C - f^T u(x)) \geq 0, \quad \forall \xi \in \mathbb{R}^+, \end{aligned}$$

and hence, $l_*^c(x, u(x), \Gamma_w) \geq 0, \forall \Gamma_w$. Now suppose that x is such that its compliance is infeasible, i.e., $f^T u(x) > C$. We will show that there exists $\Gamma_w(x) = (\alpha(x), \xi(x), \delta(x), \zeta(x), \psi(x))$, such that (41) holds, and one particular choice of vectors satisfying (41), is given by

$$\begin{aligned} \alpha(x) &= u(x), & \zeta(x) &= -\nabla_x K(x)u(x), \\ \xi(x) &= 1, & \psi(x) &= 0, \\ \delta(x) &= f - u(x)^T K_0. \end{aligned} \quad (44)$$

If now we make the change of variables: $z = x + \Delta z$, $v = u(x) + \Delta v$, and use that the stiffness matrix $K(z)$ is affine (i.e., $K(x + \Delta z) = K(x) + \nabla_x K(x)\Delta z$), then we can simplify, group common terms, and obtain

$$\begin{aligned} l_*^c(x, u(x), \Gamma_w) &= \xi (C - f^T u(x)) + \\ &\quad \sup_{\Delta v \in \mathbb{R}^d, \Delta z \in \mathbb{R}^n} \{ \Delta z \{ \alpha^T \nabla K(x)u(x) + \zeta^T u(x) + \psi \} \\ &\quad + \Delta v \{ \alpha^T K(x) + \delta^T + \zeta^T x - \xi f \} \\ &\quad + \Delta v \Delta z \{ \alpha^T \nabla K(x) + \zeta \} \}. \end{aligned}$$

In order to have that $l_*^c(x, u(x), \Gamma_w) < +\infty$, the Lagrange multipliers must satisfy

$$\begin{aligned} \alpha^T \nabla K(x)u(x) + \zeta^T u(x) + \psi &= 0, \\ \alpha^T K(x) + \delta^T + \zeta^T x &= \xi f, \\ \alpha^T \nabla K(x) + \zeta &= 0. \end{aligned}$$

This is obtained, in particular, when using the vectors given by (44). We have therefore that

$$l_*^c(x, u(x), \Gamma_w(x)) = C - f^T u(x) < 0.$$

□

There is also a property for the linearity in x , and the explicit representation of the l_*^c -functions.

Property 2 The function l_*^c defined by (42) is linear in x , and has an explicit representation, when the arguments are the Lagrange multipliers given by (44). This explicit representation is given by

$$l_*(x, u^k, \Gamma_w^k) := l_*^c(x, u^k, \zeta^k) := C - f^T u^k + \zeta^{kT} u^k [x^k - x].$$

Note the simplified notation $l_*^c(x, u^k, \zeta^k)$ that has been introduced here. For practical reasons, this notation is better suited than $l_*(x, u^k, \Gamma_w^k)$, and will therefore be used in the rest of the article.

This last property allows us to formulate a master problem for the minimum weight problem (3), including cuts to prevent designs, for which the compliance constraint is not satisfied. The relaxed master problem for the minimum weight problem (3) is given by

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\ & \text{subject to} && l_w^*(x) \leq y, \\ & && l_*^c(x, u^k, \zeta^k) \geq 0, \quad \forall k = 1, \dots, N, \\ & && Ax \leq b, \\ & && x \in \{0, 1\}^n. \end{aligned}$$

Considering the explicit representation of the l -functions, we can get the following explicit representation for the relaxed master problem

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\ & \text{subject to} && \rho^T x \leq y, \\ & && \zeta^{kT} u^k x \leq C - f^T u^k + \zeta^{kT} u^k x^k, \quad \forall k = 1, \dots, N, \\ & && Ax \leq b, \\ & && x \in \{0, 1\}^n. \end{aligned} \tag{P_w}$$

6 Finite Convergence of the Method

The proof of finite convergence of the Genralized Benders' Decomposition (GBD) method for the minimum compliance problem (2) is presented in this Section. We consider the single load case, and the assumptions (A-1) – (A-4).

Theorem 2 (Finite Convergence). *Consider the design space $X_D = \{0, 1\}^n$ and the assumptions (A-1) – (A-4). The GBD algorithm (Algorithm 1 in Section 4), for the single load minimum compliance problem (2), terminates in a finite number of iterations for any given convergence tolerance $\epsilon \geq 0$.*

Proof. The proof is similar to the proof of Theorem 2.4 in [15]. If problem (2) is infeasible, the master problem (40) will exit with an infeasibility flag in the first iteration, stopping the algorithm.

Otherwise, the sequence $\{x^k, y^k\}_k$ of solutions of the relaxed master problem will satisfy the stopping criterion, at latest when two elements of this sequence have their 0 – 1 variables identical (i.e. $x^m = x^n$, $m \neq n$). First, fix the convergence tolerance $\epsilon \geq 0$ arbitrarily. Consider (x^m, y^m) being the solution of the relaxed master problem at iteration m .

1. Solve the equilibrium equations $K(x^m)u = f$, obtaining the displacement field u^m . Its compliance is given by $f^T u^m$, satisfying $f^T u^m < \infty$. Let UB^m be the current best compliance found until iteration m . Then, we necessarily have

$$UB^m \leq f^T u^m.$$

On the other hand, as shown in Proposition 1, there exist an optimal multiplier vector ν^m and a solution u^m of the equilibrium equations. It is then possible to add the cut $l_c^*(x, u^m, \nu^m) \leq y$ to the master problem.

Let y_m^* be the optimal value of the master problem at iteration m . Then, y_m^* is a lower bound of the optimal solution of the minimum compliance problem (2). The sequence $\{y_k^*\}$ is monotone non decreasing, i.e., $y_k^* \leq y_l^*$, for $k \leq l$. Suppose now that in a future iteration of the algorithm (say iteration $m + n_0$, where n_0 is a positive integer number), the pair (x^m, y^{m+n_0}) turns out to be the solution of the master problem, i.e., $x^{m+n_0} = x^m$. Since $(x^{m+n_0}, y^{m+n_0}) = (x^m, y^{m+n_0})$ is optimal for the master problem at iteration $m + n_0$, then (x^m, y^{m+n_0}) must, in particular, satisfy:

$$l_c^*(x^m, u^m, \nu^m) \leq y_m \leq y_{m+n_0}.$$

Then, the strong duality result (Proposition 1) states that

$$l_c^*(x^m, u^m, \nu^m) = f^T u^m,$$

where $f^T u^m$ is the compliance at the iteration m . We have the set of inequalities:

$$\begin{aligned} UB^m &\leq f^T u^m = l_c^*(x^m, u^m, \nu^m) \leq y_m \leq y_{m+n_0} \\ \iff UB^m &\leq y_{m+n_0}. \end{aligned}$$

This last inequality implies that the stopping criterion is satisfied, even for $\epsilon = 0$. In summary, we have that the sequence $\{x^k, y^k\}_k$ ($(x^k, y^k) \in \{0, 1\}^n \times \mathbb{R}$) will satisfy the stopping criterion whenever $x^k = x^l$, $k \neq l$. This fact, combined with the finiteness of the design space $\{0, 1\}^n$, implies that the stopping criterion will be reached in a finite number of steps. Since m is arbitrary, this finishes the proof. \square

7 Convergence to Global Optima of GBD

In this section, we prove convergence to a global optimum of the presented Generalized Benders' Decomposition (GBD) method applied to the problems (2) and (3). Previously, several authors have shown convergence results for GBD methods applied to particular classes of nonlinear mixed-integer optimization problems. Convergence to global optima of the GBD for a class of problems, where the objective and constraints are convex in the continuous variables and linear in the discrete variables is shown in [28]. A general study of the convergence properties of the GBD method, and in particular valid arguments for the convergence to global optima, were presented in [23], when a continuous relaxation of the projected problem on the discrete variables is convex.

Theorem 3 (Convergence to Global Optimum). *If the assumptions (A-1) – (A-4) hold, then the GBD algorithm (Algorithm 1 in Section 4) applied to the minimum compliance problem (2), and adapted to the minimum weight problem (3), converges to a global optimum in a finite number of steps.*

Proof. We show the proof for the minimum compliance problem (2). The proof for the minimum weight problem (3) is analogous.

In the previous section it was proved that the GBD method converges in a finite number of steps N . Denote the best design obtained by the GBD algorithm by x^* , its compliance

value $w(x^*)$, and an exact optimal value of the master problem (40) at convergence by y^* . This means that (x^*, y^*) is an exact optimal solution of the program

$$\begin{aligned} y^* = & \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\ & \text{subject to} && - \sum_{j=1}^n x_j u^k K_j u^k - y \leq -2f^T u^k, \quad \forall k = 1, \dots, N. \\ & && x \in X, \end{aligned} \quad (P^*)$$

where X is defined by (4). We need to prove that $w(x^*)$ is a ϵ -global optimum for the mixed 0 – 1 minimum compliance problem (2). The convergence of the GBD algorithm implies that

$$w(x^*) - y^* < \epsilon,$$

and that y^* satisfies the system of inequalities

$$- \sum_{j=1}^n x_j u^k K_j u^k + f^T u^k + \sum_{j=1}^n x_j^k u^k K_j u^k \leq y^*, \quad \forall k = 1, \dots, N. \quad (45)$$

It is known that $-\sum_{j=1}^n x_j u(x')^T K_j u(x') = \nabla_x w(x')^T x$ for an arbitrary $x' \in X_M$ (see for example [7]). Then (45) can be rewritten as

$$\nabla_x w(x^k)^T (x - x^k) + w(x^k) \leq y^*, \quad \forall k = 1, \dots, N.$$

Suppose now that x^* is not a ϵ -global optimum of the minimum compliance problem (2). This means that there exists $x^{**} \in X$, such that

$$w(x^{**}) < w(x^*) - \epsilon \text{ and } w(x^{**}) < y^*. \quad (46)$$

Since the continuous relaxation of the projection of the compliance on the x -space, $w(x)$ is a convex function (see [25]), it satisfies

$$w(x^{**}) \geq w(x) + \nabla_x w(x)^T (x^{**} - x), \quad \forall x \in [0, 1]^n.$$

In particular, this last condition is satisfied by the finite set of feasible designs $\{x^k, k = 1, \dots, N\}$, obtained by the GBD algorithm

$$w(x^{**}) \geq w(x^k) + \nabla_x w(x^k)^T (x^{**} - x^k), \quad \forall k = 1, \dots, N,$$

and we have that the pair $(x^{**}, w(x^{**}))$ satisfies all the constraints of the program (P^*) . Therefore, $(x^{**}, w(x^{**}))$ is a feasible point to the program (P^*) , and hence it must satisfy $w(x^{**}) \geq y^*$, which is a contradiction to (46). This completes the proof. \square

8 The Outer-Approximation Method

In this section, we briefly present the application of the Outer-Approximation method (OA, see [11], [12, 13], and [21]) to the minimum compliance problem (2). We are interested in how OA is related to the presented Generalized Benders' Decomposition (GBD) method, within in the context of this particular class of optimization problems. These two techniques have been linked and compared for mixed-integer problems in some studies, see e.g., [11]. The theory behind both methods rely mainly on the projection theory. The OA method relies on the representation of convex sets by a collection of supporting planes, while GBD uses duality theory to generate supporting planes. The OA method requires almost the same assumptions as GBD, except that it requires in addition, convexity of

the objective and constraint functions, and compactness of the domain related to the continuous variables. As a consequence, in order to apply this technique to the minimum compliance problem (2), it is necessary to modify its formulation to satisfy the convexity requirement. The reformulation of the minimum compliance problem (2) we use, is the following

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\ & \text{subject to} && f^T K(x)^{-1} f \leq y, \\ & && x \in X, \\ & && y \in [f_L, f_U], \end{aligned} \tag{47}$$

where f_L, f_U are valid finite bounds for the objective function, and X is defined by (4). Furthermore, OA requires some constraint qualifications to hold. In this case, problem (47) satisfies some constraint qualifications for a fixed x . In addition, the function

$$C(x, y) : [0, 1]^n \times \mathbb{R} \longrightarrow \mathbb{R}, \quad C(x, y) = f^T K(x)^{-1} f - y,$$

is a convex function ([25]). Nevertheless, this formulation requires the existence of the inverse of the stiffness matrix. Therefore, the assumption (A-1) must consider only positive definite stiffness matrices, i.e., $K_0 \succ 0$ ($\eta_s > 0$), and no further generalization can be done. The OA method is based on the representation of the feasible set by a collection of supporting planes. Thus, the feasible set of the problem defined by (47) can be represented in an equivalent way as

$$\begin{aligned} & f^T K(x^k)^{-1} f + \nabla_x (f^T K(x^k)^{-1} f)^T [x - x^k] \leq y, \quad \forall x^k \in X, \\ & x \in X, \\ & y \in [f_L, f_U]. \end{aligned} \tag{48}$$

The gradient of the function $C(x, y) = f^T K(x)^{-1} f - y$ is given by $\frac{\partial}{\partial x} C = \frac{\partial}{\partial x} (f^T K(x)^{-1} f) = -u(x^k)^T \frac{\partial}{\partial x} K(x^k) u(x^k)$, where $u(x^k) = K(x^k)^{-1} f$, and $\frac{\partial}{\partial y} C = -1$, see, for example [7]. Using this result, and the representation (48), we can formulate the following optimization program

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\ & \text{subject to} && d(x^k)^T x - y \leq -f^T u(x^k) + d(x^k)^T x^k, \quad \forall x^k \in X, \\ & && x \in X, \\ & && y \in [f_L, f_U], \end{aligned} \tag{P_O}$$

where $d(x^k) = \frac{\partial C(x^k, y)}{\partial x} = \left(-u^{kT} K_1 u^k \quad -u^{kT} K_2 u^k \quad \dots \quad -u^{kT} K_n u^k \right)^T$. We have the following result.

Theorem 4. *Program (P_O) is equivalent to the minimum compliance problem (47), in the sense that (x^*, y^*) solves (47), if and only if it solves (P_O).*

Proof. This theorem is a corollary of Theorem 1 in [21], applied to the program (47). \square

We finish by comparing the formulation of the OA problem (P_O), with the master problem of the GBD (P_c) in the re-enforcement case. We see that the cuts in (P_O) are exactly the same as the ones in (P_c). However, we can note two differences. The first one is the condition $y \in [f_L, f_U]$ in (P_O), which is a necessary condition in order to have compactness in the feasible set for the variable y , and be able to use Theorem 1. In practice, this condition is unnecessary, since the function $y : \{0, 1\}^n \longrightarrow \mathbb{R}$,

$$y(z) = f^T K(x^k)^{-1} f + \nabla_x (f^T K(x^k)^{-1} f)^T [z - x^k],$$

is bounded from below. Therefore, $y = y(z)$ is bounded from below as well. The bound from above is not necessary either, since (P_O) is a minimization program. The second difference is the number of cuts included. (P_O) includes one cut per feasible design x^k , which corresponds to a finite number of cuts, but it is still too many for numerical purposes. This situation is solved in the same way as in the GBD method, i.e., by a relaxation process. After the relaxation process, we obtain an algorithm which is equivalent to the GBD algorithm (Algorithm 1 in Section 4).

9 GBD and Semidefinite Programming

In this section, we briefly present the application of a SemiDefinite Programming (SDP) formulation of the minimum compliance problem (2). For details about semidefinite programming, see for example [4], [10], [18], and [26]. Previously, there have been a number of studies about the application of semidefinite programming to minimum compliance problems with continuous design variables, see e.g. [1] and [5]. We show a clear relationship between the Generalized Benders' Decomposition (GBD), and a semidefinite programming formulation for the 0 – 1 minimum compliance problem (2). We keep all the assumptions stated previously, except that assumption (A-1) is considered with $\eta_s \geq 0$. Therefore, we treat here the pure topology case. We start by restating an important result from [1], about the compliance function and the equilibrium equations.

Proposition 4. *Let $x \in \mathbb{R}^n$, $x \geq 0$, and $y \in \mathbb{R}$ be fixed. There exists $u \in \mathbb{R}^d$ satisfying*

$$K(x)u = f \quad \text{and} \quad f^T u \leq y,$$

if and only if

$$\begin{pmatrix} y & -f^T \\ -f & K(x) \end{pmatrix} \succeq 0.$$

In particular, Proposition 4 is valid if $x \in \{0, 1\}^n$. This allows us to equivalently reformulate the minimum compliance problem (2), as the mixed 0 – 1 semidefinite program

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\ & \text{subject to} && \begin{pmatrix} y & -f^T \\ -f & K(x) \end{pmatrix} \succeq 0, \\ & && \rho^T x \leq M, \\ & && Ax \leq b, \\ & && x \in \{0, 1\}^n. \end{aligned} \tag{SDP-1}$$

The condition of semi-positiveness of the matrix in the program (SDP-1)

$$\begin{pmatrix} y & -f^T \\ -f & K(x) \end{pmatrix} \succeq 0$$

can be expressed in an equivalent way as

$$\begin{aligned} & \begin{pmatrix} \alpha \\ v \end{pmatrix}^T \begin{pmatrix} y & -f^T \\ -f & K(x) \end{pmatrix} \begin{pmatrix} \alpha \\ v \end{pmatrix} \geq 0 \quad \forall \alpha \in \mathbb{R}, v \in \mathbb{R}^d \\ & = \alpha^2 y - 2\alpha v^T f + v^T K(x) v \geq 0 \quad \forall \alpha \in \mathbb{R}, v \in \mathbb{R}^d. \end{aligned}$$

Without any loss of generality, we can consider only the case $\alpha = 1$ (this can be shown, by separately considering the cases $\alpha = 0$ and $\alpha \neq 0$), in which case it becomes

$$\begin{aligned} y - 2v^T f + v^T K(x)v &\geq 0, \quad \forall v \in \mathbb{R}^d \\ \iff -v^T K(x)v - y &\leq -2v^T f, \quad \forall v \in \mathbb{R}^d. \end{aligned}$$

The mixed 0 – 1 formulation (SDP-1) of the minimum compliance problem (2), can therefore be rewritten as

$$\begin{aligned} &\underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\ &\text{subject to} && -v^T K(x)v - y \leq -2v^T f, \quad \forall v \in \mathbb{R}^d, \\ & && \rho^T x \leq M, \\ & && Ax \leq b, \\ & && x \in \{0, 1\}^n. \end{aligned} \tag{SDP-2}$$

Note that (SDP-2) is a semi-infinite linear mixed 0 – 1 program. Many of the constraints can be discarded, obtaining as a result, a finite mixed 0 – 1 linear system. The constraints in the (SDP-2) formulation includes

$$-v^T K(x)v - y \leq -2v^T f, \quad \forall v \in \mathbb{R}^d, \tag{49}$$

which are equivalent to

$$G_x(v) \leq y, \quad \forall v \in \mathbb{R}^d, \tag{50}$$

where $G_x : \mathbb{R}^n \rightarrow \mathbb{R}$ is the concave quadratic function

$$G_x(v) = -v^T K(x)v + 2v^T f.$$

If $x^k \in \{0, 1\}^n$ is such that the function $G_{x^k}(v)$ has a maximizer, then we can replace the set of constraints (50) by the smaller set of constraints

$$G_{x^k}(v) \leq y, \quad \forall v \in \mathbb{R}^d \text{ maximizer of } G_{x^k}(v). \tag{51}$$

Since $G_{x^k}(v)$ is a concave function, $G_{x^k}(v)$ has a maximizer, if and only if the first order optimality condition $G'_{x^k}(v) = 0$ has at least one solution. Note that $G'_{x^k}(v) = 0$ implies the equilibrium equations $K(x^k)v = f$. Then, if u^k is a solution of the equilibrium equations, we have the valid cut

$$2f^T u^k - u^{kT} K(x)u^k \leq y. \tag{52}$$

On the other hand, if x_* is such that $G_{x_*}(v)$ has no maximizers, the equilibrium equations have no solutions and the function $G_{x_*}(v)$ is unbounded. As a consequence, $\forall y \in \mathbb{R}$, there exists $v_* \in \mathbb{R}^d$, such that

$$y - 2v_*^T f + v_*^T K(x_*)v_* < 0. \tag{53}$$

If x^* is an optimal solution of the problem (2), we need to find $y = y^* \in \mathbb{R}$, $v_* \in \mathbb{R}^d$, such that (53) is still valid for x_* , but not for x^*

$$y^* - 2v_*^T f + v_*^T K(x^*)v_* \geq 0. \tag{54}$$

Following Proposition 4, a sufficient condition to have the inequality (54) holding, is that y^* satisfies $y^* \geq C(x^*)$. That is, y^* must be an upper bound for the optimal value of

the compliance (i.e., we set $y^* = UB$, where UB denotes an upper bound for the optimal compliance). If $\lambda_* \in \mathbb{R}^d$ is a solution of the feasibility program

$$\begin{aligned} H_2(x_*) = \quad & \underset{\lambda \in \mathbb{R}^d}{\text{minimize}} \quad 0^T \lambda \\ & \text{subject to} \quad -2f^T \lambda + \lambda^T K(x_*) \lambda < 0, \end{aligned}$$

then the couple $y^* = UB$, $v_* = \lambda_*$ creates a valid feasibility cut. In other words, these results imply that we can exclude the design x_* from the feasible set of the master problem, by including the constraint

$$UB - 2\lambda_*^T f + \lambda_*^T K(x) \lambda_* \geq 0. \quad (55)$$

Constraints of the type of (55) will be called SDP feasibility cuts or simply SDP cuts. Considering that there could eventually exist infinite maximizers in (51), as well as infinite solutions for the program $H_2(x)$, we have still potentially an infinite number of SDP cuts to include (v_* and y_* are not necessarily unique). It is possible to prove that it is enough to consider only one SDP cut per design. First, if $G_{x^k}(v)$ has one or several maxima, any of them produces a valid constraint as (52), preventing x^k from being the 0 – 1 part of an optimal solution at the next master problem (unless the stopping criterion is satisfied and x^k is the optimal solution obtained by the algorithm). The inclusion of other maxima leads to alternative optimality constraints and could potentially speed up the convergence of the algorithm, but they are not necessary to guarantee convergence. Second, if $G_{x^k}(v)$ has no maxima, then x^k is an infeasible design. It is enough to find one λ_* , solution of $H_2(x_*)$, to assure that the SDP cut (55) excludes x_* from the feasible set of the master program.

In summary, the infinite system of constraints (49) is equivalent to the finite set of constraints

$$\begin{aligned} G_x(u^k) \leq y, \quad & u^k \in \mathbb{R}^d \text{ satisfies } K(x^k)u^k - f = 0 \\ UB - G_x(\lambda^k) \geq 0, \quad & \text{if } K(x^k)v - f = 0 \text{ has no solutions,} \end{aligned} \quad (56)$$

with λ^k a solution of the program $H_2(x^k)$. After the equivalence between the sets of constraints (49) and (56), we can state a finitely constrained program, equivalent to (SDP-2), given by

$$\begin{aligned} \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} \quad & y \\ \text{subject to} \quad & -u^k{}^T K(x)u^k - y \leq -2f^T u^k, \quad K(x^k)v - f = 0 \text{ feasible,} \\ & -\lambda^k{}^T K(x)\lambda^k \leq UB - 2f^T \lambda^k, \quad K(x^k)v - f = 0 \text{ infeasible,} \quad (\text{SDP-3}) \\ & \rho^T x \leq M, \\ & Ax \leq b, \\ & x \in \{0, 1\}^n. \end{aligned}$$

Problem (SDP-3) provides a basis for an alternative algorithm to solve the minimum compliance problem (2). This algorithm should be tested in numerical examples. In the case of the minimum weight problem (3), only minor changes must be done to formulate the corresponding semidefinite programming problem.

10 Final Remarks

The proposed Generalized Benders' Decomposition (GBD) method has been applied to minimum compliance and minimum weight topology optimization problems, considering

a single load condition and a re-enforcement scenario. It was proven that the method converges to a global optimum in a finite number of iterations. This is mainly a consequence of two conditions. First, the convexity of the compliance as a function of the design variables and second, a strong duality property holding for the subproblem, even though this subproblem is not convex. The method has been linked to, and compared with the Outer-Approximation method and Semi-Definite Programming (SDP) formulations of the considered problems. In particular, GBD and OA generate the same optimality cuts in the re-enforcement situation. The SDP formulations generate slightly different cuts, which can be used to accelerate the rate of convergence in numerical computations. The algorithm has been explicitly formulated and numerical experiments will indicate the practical skills of the algorithm. In addition, the generalization to pure topology optimization problems with multiple load conditions are natural steps to follow the theoretical investigation presented herein. These generalizations and numerical experiments are presented in the second part of this article ([22]).

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References

- [1] W. Aichtziger and M. Kočvara. Structural topology optimization with eigenvalues. *SIAM Journal on Optimization*, 18(4):1129–1164, 2007.
- [2] W. Aichtziger and M. Stolpe. Global optimization of truss topology with discrete bar areas - Part I: Theory of relaxed problems. *Computational Optimization and Applications*, (40):247–280, 2007.
- [3] W. Aichtziger and M. Stolpe. Truss topology optimization with discrete design variables - guaranteed global optimality and benchmark examples. *Structural and Multidisciplinary Optimization*, 34(1):1–20, 2007.
- [4] M.F. Anjos and A. Vannelli. Computing globally optimal solutions for single-row layout problems using semidefinite programming and cutting planes. *INFORMS Journal on Computing*, 20(4):611–617, 2008.
- [5] A. Ben-Tal and A. Nemirowski. Robust truss topology design via semidefinite programming. *SIAM Journal on Optimization*, 7(4):991–1016, 1997.
- [6] J.F. Benders. Partitioning procedures for solving mixed-variables programming problems. *Numerische Mathematik*, 4(1):238–252, 1962.
- [7] M.P. Bendsøe and O. Sigmund. *Topology Optimization: Theory, Methods and Applications*. Springer-Verlag, 2nd edition, 2003.
- [8] S. Bollapragada, O. Ghattas, and J.N. Hooker. Optimal design of truss structures by logic-based branch and cut. *Operations Research*, 49(1):42–51, 2001.
- [9] S.C. Brenner and L.R. Scott. *The mathematical theory of finite element methods*. Springer, 1994.

- [10] B. Craven and B. Mond. Linear programming with matrix variables. *Linear Algebra and its Applications*, 38:73–80, 1981.
- [11] M.A. Duran and I.E. Grossmann. An outer-approximation algorithm for a class of mixed-integer nonlinear programs. *Mathematical Programming*, 36(3):307–339, 1986.
- [12] A.M. Geoffrion. Elements of large-scale mathematical programming: Part I: Concepts. *Institute of Management Sciences*, 16(11):652–675, 1970.
- [13] A.M. Geoffrion. Elements of large-scale mathematical programming: Part II: Synthesis of Algorithms and Bibliography. *Institute of Management Sciences*, 16(11):676–691, 1970.
- [14] A.M. Geoffrion. Duality in non linear programming: a simplified applications-oriented development. *SIAM Review*, 13(1):1–37, 1971.
- [15] A.M. Geoffrion. Generalized Benders decomposition. *Journal of Optimization Theory and Applications*, 10(4):237–260, 1972.
- [16] O.K. Gupta and A. Ravindran. Branch and bound experiments in convex nonlinear integer programming. *Management Science*, 31(12):1533–1546, 1985.
- [17] J. Hiriart-Urruty and C. Lemaréchal. *Convex Analysis and Minimization Algorithms II*. Springer-Verlag, 1996.
- [18] M. Kojima, S. Kojima, and S. Hara. Linear algebra for semidefinite programming. Technical report, Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, 1994. Research Report B-290.
- [19] A.H. Land and A.G. Doig. An automatic method of solving discrete programming problems. *Econometrica: Journal of the Econometric Society*, 28(3):497–520, 1960.
- [20] R. Lazimy. Extension of the generalized Benders decomposition. *Communications in Applied Numerical Methods*, 6:195–203, 1986.
- [21] S. Leyffer and R. Fletcher. Solving mixed-integer nonlinear programs by outer-approximation. *Mathematical Programming*, 66(3):327–349, 1994.
- [22] E. Munoz and M. Stolpe. Benders’ decomposition for topology optimization problems – Part 2: Extensions and numerical experiments. Technical report, Department of Mathematics, Technical University of Denmark, 2009.
- [23] N.V. Sahinidis and I.E. Grossman. Convergence properties of generalized Benders decomposition. *Computers and Chemical Engineering*, 15:481–491, 1991.
- [24] R.A. Stubbs and S. Mehrotra. Generating convex polynomial inequalities for mixed 0–1 programs. *Journal of Global Optimization*, 24(3):311–332, 2002.
- [25] K. Svanberg. On the convexity and concavity of compliances. *Structural and Multidisciplinary Optimization*, 7(1-2):42–46, 1994.
- [26] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38(1):49–95, 1996.
- [27] L.A. Wolsey and G.L. Nemhauser. *Integer and combinatorial optimization*. Wiley-Interscience, 1998.

- [28] Y. Zhu and T. Kuno. Global optimization of non-convex MILP by a hybrid branch-and-bound and revised general Benders decomposition approach. *Industrial and Engineering Chemistry Research*, (42):528–539, 2003.

Article 2

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Generalized Benders' Decomposition for Topology Optimization Problems – Part 2: Extensions and Numerical Experiments

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Abstract

This two part article considers the non-linear mixed-integer optimization programs that appear in structural topology optimization. The main objective of this work is to present a generalized Benders' decomposition (GBD) method for solving single and multiple load minimum compliance (maximum stiffness) and minimum weight problems to global optimality. The GBD technique is a classical method for mixed-integer programming, in which the non-linear mixed-integer program is replaced by a sequence of linear mixed-integer programs.

Part one presents the theoretical aspects of the method, including a proof of finite convergence and conditions for obtaining global optimal solutions to single load re-enforcement problems, i.e. structural optimization problems for which no holes can be created. The method is also linked to, and compared with the Outer-Approximation approach and 0 – 1 semidefinite programming formulations of the considered problems.

In the second part, the method and the theoretical results are generalized to pure topology optimization problems under multiple load conditions. Several ways to accelerate the method are suggested and an implementation is described in detail. Finally, a set of truss topology optimization problems are numerically solved to global optimality.

Mathematical Subject Classification (2000): 90C90, 74P05, 74P15

Keywords: Structural Topology Optimization, Global Optimization, Generalized Benders' Decomposition.

1 Introduction

We consider structural topology optimization problems with discrete design variables, in particular classical minimum compliance (maximum stiffness) and minimum weight problems. We study the mentioned problems in non-linear mixed 0 – 1 formulations, both from the theoretical and numerical points of view. The theoretical properties of a generalized Benders' decomposition (GBD) method applied to re-enforcement problems were presented in the first part of this article ([16]). The GBD technique is based on solving a sequence of linear mixed-integer optimization programs (relaxed master problems),

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that approximate the original non-linear mixed-integer program. These programs can be solved by any method for linear mixed-integer optimization, as for example, branch-and-cut methods (see e.g. [20] and [25]), or branch-and-bound (see e.g. [10] and [13]). The GBD algorithm presented in [16] for single load minimum compliance problems forms a monotone sequence of lower and upper bounds of the optimal value. This sequence and the corresponding sequence of optimal solutions to the relaxed GBD master problem, were proven to converge in a finite number of iterations to the optimal value and a globally optimal design, respectively.

In this part of the article, we present generalizations of the method to pure topology optimization problems under multiple load conditions, i.e. problems including real void as a candidate material. We also present an implementation of the method and suggest several techniques to improve the rate of convergence. Finally, we report on the numerical experience obtained with the method, when applied to the design of two-dimensional truss structures.

This second part of the article is organized as follows. Section 2 restates the mixed 0 – 1 formulation of the considered minimum compliance and minimum weight problems, as well as the assumptions considered in this part of the article. Section 3 presents the extension of the GBD presented in the first part to pure topology optimization problems and to problems with multiple load cases. Section 4 states the GBD method for multiple load minimum weight topology optimization problems. In Section 5, we suggest several techniques to accelerate the numerical performance of GBD method applied to topology optimization problems. Section 6 describes the implementation of the method used in the numerical experiments. Section 7 presents the numerical examples for the problems treated in this article, and finally Section 8 contains a brief discussion of the presented results and an outline of the future work in this research field that will be carried out.

2 Problem Statement

For completeness, and to make this part of the article self-contained, we repeat most of the section Problem Statement from the first part of the article ([16]).

We consider a closed-bounded design domain $\Omega_c \subset \mathbb{R}^2$ or $\Omega_c \subset \mathbb{R}^3$, with $\partial\Omega_c$ of class \mathcal{C}^1 . The design space related to Ω_c corresponds to the power set $P(\Omega_c)$ of Ω_c (i.e., the set of all subsets of Ω_c). $P(\Omega_c)$ has infinite cardinality. For numerical purposes, the design space Ω_c is discretized into a finite set of n small design regions, $\Omega_D = \{D_1, \dots, D_n\}$, with $\cup_{i=1}^n D_i = \Omega_c$, $D_i \cap D_j = \emptyset$, if $i \neq j$. A 0 – 1 design variable is linked to each D_i . The discrete design space corresponds now to the power set of Ω_D , $P(\Omega_D)$, of cardinality 2^n . Typically (but not necessarily), these design regions coincide with the finite elements that are used to compute the response (displacements, stresses, etc.) of a structure under loading. The design variables can represent thicknesses, areas, or densities of a given isotropic design material in each of the elements. They can also, as in the case of design of composite structures, represent a choice of available composite materials. A design is characterized by a vector element x in the discrete design set $\chi_D = \{0, 1\}^n$, where n is the number of design variables induced by the discretization. In order to ensure that the problem is mathematically well formulated, we assume that suitable boundary conditions and external loads are imposed. The design problem consists in finding the optimal vector $x^* \in \chi_D$ that minimizes the objective function (compliance, weight, etc.), under some restrictions in the design space χ_D . Throughout, we consider optimal design of mechanical structures in linear elasticity, subjected to static forces. We assume that the response of the structure is computed using the finite element method (see e.g. [3]).

The vector $x \in \chi_D$ will throughout be denoted the design variable vector, or simply

the design variable. The elastic equilibrium equations relating an external force f applied to the structure and the corresponding displacement u are given by

$$K(x)u = f. \quad (1)$$

Here, $u \in \mathbb{R}^d$ is the vector of displacements, and $f \in \mathbb{R}^d$ represents a given external static load for each of the d degrees of freedom introduced by the discretization of the design space. The stiffness matrix $K(x) \in \mathbb{R}^{d \times d}$ is a function of the design variable x , and we assume that the load vector f is independent of the design x .

The first of the problems we study is the minimum compliance (maximum stiffness) problem, which is formulated as

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, u \in \mathbb{R}^d}{\text{minimize}} && f^T u \\ & \text{subject to} && K(x)u = f, \\ & && \rho^T x \leq M, \\ & && Ax \leq b, \\ & && x \in \chi_D = \{0, 1\}^n. \end{aligned} \quad (2)$$

In (2), $M > 0$ is the maximum amount of material allowed for the structure, and $\rho \in \mathbb{R}^n$ is the vector of material densities. The constraints $Ax \leq b$ state a general set of linear inequalities involving only the design variable. The second problem formulation we study, is the minimum weight problem

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, u \in \mathbb{R}^d}{\text{minimize}} && \rho^T x, \\ & \text{subject to} && K(x)u = f, \\ & && f^T u \leq C, \\ & && Ax \leq b, \\ & && x \in \chi_D = \{0, 1\}^n, \end{aligned} \quad (3)$$

where $C > 0$ is the maximum allowed compliance for the structure. Problems (2) and (3) are both non-linear mixed 0 – 1 programs. They are also non-convex problems, due to the integer nature of the design variables, and the non-linearity of the equilibrium equations (1).

Denote the sets

2.1 Assumptions

A summary of the important assumptions we consider throughout this article is the following:

- (A-1) The stiffness matrix $K(x)$ is symmetric, affine in x , and positive semi definite for all $x \in \{0, 1\}^n$. The matrix $K(x)$ is given by

$$K(x) = K_0 + \sum_{j=1}^n x_j K_j, \quad (4)$$

where $K_j \in \mathbb{R}^{d \times d}$ is the symmetric positive semi definite element stiffness matrix for the j -th design variable, and $K_0 \in \mathbb{R}^{d \times d}$ is a given symmetric positive semidefinite matrix ($K_0 \succeq 0$). We assume that

$$K_0 = \eta_s \sum_{j=1}^n K_j, \quad (5)$$

with $0 \leq \eta_s \ll 1$. In re-enforcement problems, we have $\eta_s > 0$, while $\eta_s = 0$ in pure topology optimization problems. η_s will be called the re-enforcement parameter.

- (A-2) The compliance and mass limits, C and M respectively, satisfy $C > 0$ and $0 < M < \sum_{j=1}^n \rho_j$, where $\rho_j \geq 0$ for all $j = 1, \dots, n$.
- (A-3) The external load $f \in \mathbb{R}^d \setminus \{0\}$.
- (A-4) There are no special assumptions on A and b , except for the requirement that the feasible set related to the constraint $Ax \leq b$ is non empty.

Assumptions (A-2) – (A-4) are identical to the corresponding assumptions in the first part of the article [16]. The assumption (A-1) generalizes the first assumption in [16], since it allows holes to be introduced in the structure.

3 Extensions of the Design Problem

In the first part of the article ([16]), we presented a Generalized Benders' Decomposition (GBD) method for the single load minimum compliance and minimum weight problems (2) and (3), respectively. We considered the re-enforcement case, i.e. $\eta_s > 0$ in equation (5) in assumption (A-1). Here we extend the presented method to pure topology optimization problems, i.e. $\eta_s = 0$ in equation (5) in assumption (A-1), where infeasibility of the equilibrium equations (1) may occur for a subset of the design space. Consequently, a new type of cuts preventing this type of infeasibility is introduced. These cuts are the so called GBD feasibility cuts. The second generalization presented here is the extension to the multiple load case, where the worst-case compliance (over all load conditions) is minimized. The master problems for these cases are formulated and some final remarks are presented.

In the first part of the article, the relaxed GBD master problem for the single load minimum compliance problem (2), was stated as

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\ & \text{subject to} && l_c^*(x, u^k, \nu^k) \leq y, \quad \forall k = 1, \dots, N, \\ & && \rho^T x \leq M, \\ & && Ax \leq b, \\ & && x \in \{0, 1\}^n. \end{aligned} \tag{6}$$

The relaxed GBD master problem for the minimum weight problem (3), was given by

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\ & \text{subject to} && \rho^T x \leq y, \\ & && l_*^c(x, u^k, \nu^k) \geq 0, \quad \forall k = 1, \dots, N, \\ & && Ax \leq b, \\ & && x \in \{0, 1\}^n, \end{aligned} \tag{7}$$

where the functions, l_c^* and l_*^c are defined as

$$\begin{aligned} l_c^*(x, u^k, \nu^k) &= f^T u^k + \nu^{kT} u^k [x^k - x], \\ l_*^c(x, u^k, \nu^k) &= C - f^T u^k + \nu^{kT} u^k [x - x^k], \text{ with} \\ \nu^k &= \left(u^{kT} K_1 \quad u^{kT} K_2 \quad \dots \quad u^{kT} K_n \right)^T, \\ u^k &= K(x^k)^{-1} f, \end{aligned} \tag{8}$$

and x^k is the solution of the k -th relaxed master problem. These master problems are used in the investigations presented in this second part of the article.

Notation 1. *In the definition of the functions l_c^* and l_*^c given by (8), a special notation is used. This notation will be used throughout the article. The expression*

$$\nu^{kT} u^k [x^k - x] = \nu^{kT} u^k x^k - \nu^{kT} u^k x,$$

should be understood in the following way. Each product of three terms

$$\nu^T u x, \text{ with } \nu \in \mathbb{R}^{nd}, u \in \mathbb{R}^d, x \in \mathbb{R}^n$$

is interpreted as

$$\nu^T u x = \sum_{j=1}^d \sum_{k=1}^n \nu(d[k-1] + j) u(j) x(k).$$

In particular, using the expression of ν^k in (8), the $\nu^{kT} u^k x^k$ terms are interpreted throughout the article as

$$\nu^{kT} u^k x^k = \sum_{j=1}^n x_j^k u^{kT} K_j u^k.$$

3.1 Extension to General Topology Optimization Problems

The GBD method can be extended to the pure topology optimization case, where only semidefiniteness of the stiffness matrix $K(x)$ holds. This corresponds to consider the assumption (A-1), with $\eta_s = 0$ in equation (5). This means that the stiffness matrix is no longer affine, but linear in the design variables, i.e.,

$$K(x) = \sum_{j=1}^n x_j K_j.$$

In this case, the matrix $K(x)$ is in general singular. Thus, a design x^k may eventually lead to an infeasible set of equilibrium equations (1), preventing us to obtain a valid displacement field u^k . In case of infeasibility, we can not obtain a valid optimality cut (8) for the master problem (6). In this situation, GBD includes a set of feasibility cuts, which we derive in this section. We start by defining the linear least squares problem,

$$\begin{aligned} P_1(x) = & \underset{u \in \mathbb{R}^d}{\text{minimize}} && \| K(x)u - f \|_2^2 \\ & \text{subject to} && \| u \|_2 \leq C_{ls}, \end{aligned} \quad (9)$$

where $C_{ls} > 0$ is a sufficiently large scalar. The following result is necessary to define the feasibility cuts.

Proposition 1. *Consider a stiffness matrix $K(x) \succeq 0$. The equilibrium equations $K(x)u - f = 0$ have no solutions, if and only if*

$$\Phi_{\mathbb{F}}(x) := \inf_{\lambda \in \mathbb{R}^d, \phi \in \mathbb{R}^{nd}} \sup_{z \in \mathbb{R}^n, v \in \mathbb{R}^d} \{ \lambda^T (K(z)v - f) + \phi^T (vz - u_*(x)x) \} < 0, \quad (10)$$

where $u_*(x)$ is any minimum norm solution

$$u_*(x) \in \underset{v \in \mathbb{R}^d}{\operatorname{argmin}} P_1(x),$$

and $P_1(x)$ is the least squares problem defined by (9). In particular, if we define

$$l_*^f(x, u, \lambda, \phi) := \supremum_{z \in \mathbb{R}^n, v \in \mathbb{R}^d} \{ \lambda^T [K(z)v - f] + \phi^T [vz - ux] \}, \quad (11)$$

and set $\phi_* = -\lambda^T \nabla_x K(x)$, then l_*^f becomes

$$l_*^f(x, u, \lambda, \phi_*) = \lambda^T K(x)u - f^T \lambda. \quad (12)$$

Proof. To prove that such $u_*(x)$ exists, it is enough to prove that there exists a minimizer of $P_1(x)$. This can be proven by using least squares theory for over-determined linear systems in the rank-deficient case ($\operatorname{rank}(K(x)) < d$), see for example [6, 7]. In particular, the existence of a minimizer of $P_1(x)$ follows from the fact that in $P_1(x)$, a continuous function is minimized over a closed bounded set (a compact set in \mathbb{R}^d). $P_1(x)$ therefore attains its extremes.

Note that problem (10) represents the dual formulation of the feasibility problem $\mathbb{F}(x)$, given by

$$\begin{aligned} \mathbb{F}(x) = \quad & \supremum_{z \in \mathbb{R}^n, v \in \mathbb{R}^d} \quad 0^T z + 0^T v \\ & \text{subject to} \quad K(z)v - f = 0, \\ & \quad \quad \quad vz - u_*(x)x = 0. \end{aligned} \quad (13)$$

First, suppose that x is such that there exists a solution $u(x)$ of the equilibrium equations (1). It follows that since C_{ls} is large enough, $u(x)$ is at the same time, a solution of the least squares problem (9). Therefore, we can suppose that $u(x) = u_*(x)$. It is then possible to see that $(z = x, v = u(x))$ is the unique optimal solution for (13). The weak duality theorem (See [11], p. 149 or [8], p. 9) ensures that $\Phi_{\mathbb{F}}(x) \geq \mathbb{F}(x) = 0$.

Suppose now that x is such that $K(x)v - f = 0$ has no solutions, and $u_*(x)$ is a minimizer of $P_1(x)$. We can show that there exist $\lambda_* \in \mathbb{R}^d$ and $\phi_* \in \mathbb{R}^{nd}$, for which

$$l_*^f(x, u_*(x), \lambda_*, \phi_*) < 0.$$

These vectors are given explicitly by

$$\lambda_* = \lambda(x) = f - K(x)u_*(x), \quad (14)$$

$$\phi_* = \phi(x) = -\nabla_x K(x)\lambda_*. \quad (15)$$

The definitions (10) of $\Phi_{\mathbb{F}}$ and (11) of l_*^f imply the validity of the inequality

$$\Phi_{\mathbb{F}}(x) \leq l_*^f(x, u_*(x), \lambda_*, \phi_*). \quad (16)$$

We make the changes of variables $z = x + \Delta z$, $v = u_*(x) + \Delta v$, and use the fact that the stiffness matrix $K(z)$ is linear in z ($K(x + \Delta z) = K(x) + \nabla_x K(x)\Delta z$). Then, we obtain

$$\begin{aligned} l_*^f(x, u_*(x), \lambda, \phi) = \quad & \supremum_{\Delta z \in \mathbb{R}^n, \Delta v \in \mathbb{R}^d} \{ \lambda^T [(K(x) + \Delta z \nabla_x K(x))[u_*(x) + \Delta v] - f] \\ & + \phi^T [u_*(x)\Delta z + \Delta v x + \Delta v \Delta z] \}. \end{aligned}$$

Since $K(x) = \nabla_x K(x)x$, we can simplify terms

$$l_*^f(x, u_*(x), \lambda, \phi) = \sup_{\Delta z \in \mathbb{R}^n, \Delta v \in \mathbb{R}^d} \{[\phi^T + \lambda^T \nabla_x K(x)][\Delta z \Delta v + u_*(x) \Delta z + x \Delta v]\} \\ + \lambda^T [K(x)u_*(x) - f].$$

If we evaluate $\phi_* = -\nabla_x K(x)\lambda$, then the supremum term is canceled

$$l_*^f(x, u_*(x), \lambda, \phi_*) = \lambda^T K(x)u_*(x) - f^T \lambda,$$

obtaining the representation (12) as wanted. To complete the proof, we need to find λ_* such that $\lambda_*^T [K(x)u_*(x) - f] < 0$. There is an infinite number of vectors $\lambda_* \in \mathbb{R}^d$ satisfying this inequality. Take, for example $\lambda_* = f - K(x)u_*(x)$, in which case

$$l_*^f(x, u_*(x), \lambda_*, \phi_*) = [f - K(x)u_*(x)]^T [K(x)u_*(x) - f] \\ = -\|K(x)u_*(x) - f\|_2^2 < 0.$$

This last inequality, together with (16), finishes the proof. \square

Proposition 1 proves that if a given design x^k is infeasible for the equilibrium (1), we can find vectors u_*^k , λ^k , and ϕ^k , such that $l_*^f(x^k, u_*^k, \lambda^k, \phi^k) < 0$. By including the feasibility cut $l_*^f(x, u_*^k, \lambda^k, \phi^k) \geq 0$, we prevent the design x^k from being feasible in the relaxed master problem in the following iterations.

However, if x^* is a global optimal solution for the considered problem, nothing ensures that x^* satisfies this feasibility cut, i.e. we can not ensure that x^* satisfies $l_*^f(x^*, u_*^k, \lambda^k, \phi^k) \geq 0$. In order to do so, we start by recalling an equivalent reformulated version of a proposition stated and proved in [1].

Proposition 2. *Let $x \in \mathbb{R}^n$, $x \geq 0$, and $y \in \mathbb{R}$ be fixed. There exists $u \in \mathbb{R}^d$ satisfying*

$$K(x)u = f \quad \text{and} \quad f^T u \leq y,$$

if and only if

$$y - 2f^T v + v^T K(x)v \geq 0, \quad \forall v \in \mathbb{R}^d.$$

Theorem 1. *Consider $x^k \in \{0, 1\}^n$, $x^k \neq 0^n$, an infeasible design for the equilibrium equations (1), and the feasibility problem*

$$H(x^k) = \begin{aligned} &\underset{\lambda \in \mathbb{R}^d}{\text{minimize}} && 0^T \lambda \\ &\text{subject to} && \lambda^T K(x^k)\lambda - f^T \lambda < 0, \\ &&& f^T \lambda - UB \geq 0, \end{aligned} \tag{17}$$

where UB is a valid upper bound for the compliance $c(x^)$, with x^* an optimal solution of problems (2) or (3), and $\phi^k = -\nabla_x K(x^k)\lambda^k$. Then, $H(x^k)$ possesses a solution $\lambda^k \in \mathbb{R}^d$, and the following inequalities are valid*

$$l_*^f(x^k, \lambda^k, \lambda^k, \phi^k) < 0, \\ l_*^f(x^*, \lambda^k, \lambda^k, \phi^k) \geq 0.$$

Proof. We start by proving the existence of a solution for the feasibility program $H(x^k)$. We do it by a contradiction argument. First, note that the function $F_{x^k}^1 : \mathbb{R}^d \rightarrow \mathbb{R}$, $F_{x^k}^1(v) = -f^T v + v^T K(x^k)v$ is unbounded from both sides. In addition, the function $F_{x^k}^2 :$

$\mathbb{R}^d \longrightarrow \mathbb{R}_+$, $F_{x^k}^2(v) = v^T K(x^k)v$ is unbounded from above. Then, from the unboundedness of $F_{x^k}^1$, we can find $\lambda_0 \in \mathbb{R}^d$ such that

$$\lambda_0^T K(x^k) \lambda_0 - f^T \lambda_0 < 0.$$

Suppose now that the problem $H(x^k)$ is infeasible, i.e.,

$$\begin{aligned} \forall \lambda \in \mathbb{R}^d : \lambda^T K(x^k) \lambda - f^T \lambda < 0 &\implies f^T \lambda - UB < 0 \\ &\implies f^T \lambda < UB \\ &\implies \lambda^T K(x^k) \lambda < UB. \end{aligned}$$

Equivalently we can write, by contraposition

$$\forall \lambda \in \mathbb{R}^d : \lambda^T K(x^k) \lambda \geq UB \implies \lambda^T K(x^k) \lambda - f^T \lambda \geq 0. \quad (18)$$

The contradiction arises when looking at any vector $\lambda_1 \in \mathbb{R}^d$, such that $\lambda_1^T K(x^k) \lambda_1 \geq UB$. Such λ_1 exists, from the unboundedness of the function $F_{x^k}^2$. It follows that the vector $-\lambda_1$ also satisfies $(-\lambda_1)^T K(x^k) (-\lambda_1) \geq UB$, and whether λ_1 , or $-\lambda_1$ (but not both) satisfies $\lambda^T K(x^k) \lambda - f^T \lambda < 0$. This is a contradiction to (18), which proves the feasibility of the problem $H(x^k)$.

Now, if λ^k is feasible for the problem $H(x^k)$, then it satisfies the inequality $\lambda^{kT} K(x^k) \lambda^k - f^T \lambda^k < 0$, or using the representation of the l_*^f -function given in Proposition 1, with $\phi^k = -\nabla_x K(x^k) \lambda^k$, $u^k = \lambda^k$,

$$l_*^f(x^k, \lambda^k, \lambda^k, \phi^k) < 0.$$

Suppose now that we have a valid upper bound for the global optimal compliance UB (i.e., a value UB such that $f^T u(x^*) \leq UB$). Then, by taking $y = UB$ in Proposition 2, we have a valid inequality for any x^* , global optimal design of problems (2) or (3)

$$UB - 2f^T v + v^T K(x^*)v \geq 0, \quad \forall v \in \mathbb{R}^d,$$

or equivalently

$$-f^T v + v^T K(x^*)v \geq f^T v - UB, \quad \forall v \in \mathbb{R}^d.$$

Then any vector $v \in \mathbb{R}^d$ that satisfies $UB - f^T v \geq 0$, also satisfies $-f^T v + v^T K(x^*)v \geq 0$. In particular, $\lambda^{kT} K(x^*) \lambda^k - f^T \lambda^k \geq 0$, or

$$l_*^f(x^*, \lambda^k, \lambda^k, \phi^k) \geq 0.$$

□

The real importance of this result is that it provides a method to generate valid feasibility GBD cuts, preventing a set of infeasible designs to be feasible for the GBD master problem. At the same time it ensures that none of the optimal solutions is cut away from its feasible set. We have now a result, which arises as a consequence of Theorem 1.

Property 1. *The function $l_*^f(x, \lambda, \lambda, \phi)$, defined by (11) is linear in x , and has an explicit representation, when the other arguments are λ^k, ϕ^k , where λ^k is a solution of the feasibility problem (17), and $\phi^k = -\nabla_x K(x^k) \lambda^k$. This explicit representation is given by*

$$l_*^f(x, \lambda^k, \lambda^k, \phi^k) = l_*^f(x, \lambda^k, \phi^k) = -\phi^{kT} \lambda^k x - f^T \lambda^k. \quad (19)$$

Consequently, the GBD feasibility cut is given by

$$\phi^{kT} \lambda^k x \leq -f^T \lambda^k.$$

Now we can formulate a master problem, including cuts to prevent a design x , for which the equilibrium equations (1) are infeasible, to be a solution of the master problem in the following iterations. After including these l_*^f -cuts in the master problem (6), the relaxed master problem for the minimum compliance problem (2), in the case $\eta_s = 0$, becomes

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\ & \text{subject to} && l_c^*(x, u^k, \Gamma_c^k) \leq y, \quad \forall k = 1, \dots, p, \\ & && l_*^f(x, \lambda^k, \phi^k) \geq 0, \quad \forall k = 1, \dots, q, \\ & && \rho^T x \leq M, \\ & && Ax \leq b, \\ & && x \in \{0, 1\}^n. \end{aligned} \tag{P_{cf}^*}$$

Considering the explicit representation of the l_*^f -functions (19) and the l_c^* -function in (8), we can get an explicit representation of the relaxed master problem

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\ & \text{subject to} && -\nu^{kT} u^k x - y \leq -2f^T u^k, \quad k = 1, \dots, p, \\ & && \phi^{kT} \lambda^k x \leq -f^T \lambda^k, \quad k = 1, \dots, q, \\ & && \rho^T x \leq M, \\ & && Ax \leq b, \\ & && x \in \{0, 1\}^n, \end{aligned} \tag{P_{cf}}$$

with ν^k as in (8), $\phi^k = -\nabla_x K(x^k) \lambda^k$, and λ^k a solution of the feasibility problem (17). Note that we have simplified the expression for the l_c^* -cuts in the program (P_{cf}) to

$$-\nu^{kT} u^k x - y \leq -2f^T u^k, \tag{20}$$

with respect to the l_c^* -cuts (8) of the re-enforcement case. This simplification comes from the fact that when $\eta_s = 0$ in (4), we have that $\nu^{kT} u^k x^k = f^T u^k$. Note as well that the l_c^* -cuts (20) are equivalent to the SDP optimality cuts derived in the first part of the article ([16]). For the minimum weight problem (3), the formulation of the relaxed master problem with $\eta_s = 0$ is analogous.

3.2 Generalization to the Multiple Load Case

In the multiple load case problem, we would like to design an optimal structure, which is subjected to several load conditions f_1, \dots, f_m , and each of them is considered as a different and independent scenario. We modify slightly some of the assumptions from the single load case.

- (A-2) The compliance and mass limits, C_l , $l = 1, \dots, m$, and M respectively, satisfy $C_l > 0$, $\forall l = 1, \dots, m$, and $0 < M < \sum_{j=1}^n \rho_j$, where $\rho_j \geq 0$ for all $j = 1, \dots, n$.
- (A-3) Each of the loads $f_1, \dots, f_m \in \mathbb{R}^d$ is non null, i.e. $f_l \neq 0$, $\forall l = 1, \dots, m$.

For simplicity, we consider $\eta_s > 0$ in assumption (A-1). The minimum weight problem for a multiple load conditions case is formulated as

$$\begin{aligned}
& \underset{x \in \mathbb{R}^n, u_1, \dots, u_m \in \mathbb{R}^d}{\text{minimize}} && \rho^T x \\
& \text{subject to} && K(x)u_l = f_l, \quad l = 1, \dots, m, \\
& && f_l^T u_l \leq C_l, \quad l = 1, \dots, m, \\
& && Ax \leq b, \\
& && x \in \{0, 1\}^n.
\end{aligned} \tag{21}$$

This program has m equilibrium equations, one per load condition, and m compliance constraints, again, one per load condition.

For the problem (21), the GBD algorithm only includes a cut for each load condition for which a design x^k does not satisfy the corresponding compliance constraint. The relaxed master problem related to the problem (21) is given by

$$\begin{aligned}
& \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\
& \text{subject to} && \rho^T x \leq y \\
& && l_*^c(x, u_{l_j}^k, \nu_{l_j}^k) \geq 0, \quad \begin{array}{l} k = 1, \dots, N, \\ j = 1, \dots, m^k, \end{array} \\
& && Ax \leq b, \\
& && x \in \{0, 1\}^n,
\end{aligned} \tag{P_{wm}^*}$$

where $m^k \leq m$ counts the number of infeasible or active compliance constraints for the design x^k . Considering the representation of $l_*^c(x, u_{l_j}^k, \nu_{l_j}^k)$ given by (8), we can get an explicit form for the relaxed master problem for the multiple load minimum weight problem (21). It is given by

$$\begin{aligned}
& \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\
& \text{subject to} && \rho^T x \leq y \\
& && -\nu_{l_j}^{kT} u_{l_j}^k [x - x^k] \geq C_{l_j} - f_{l_j}^T u_{l_j}^k, \quad \begin{array}{l} k = 1, \dots, N, \\ j = 1, \dots, m^k, \end{array} \\
& && Ax \leq b, \\
& && x \in \{0, 1\}^n.
\end{aligned} \tag{P_{wm}}$$

Now, with respect to the minimum compliance problem, several possibilities for the objective function are available. For example, a worst case scenario for the compliance. In this case, the objective function becomes

$$F(u_1, \dots, u_m) = \max_{1 \leq l \leq m} \{f_l^T u_l\}.$$

The minimum compliance problem for the multiple loads case is given by

$$\begin{aligned}
& \underset{x \in \mathbb{R}^n, u_1, \dots, u_m \in \mathbb{R}^d}{\text{minimize}} && \max_{1 \leq l \leq m} \{f_l^T u_l\} \\
& \text{subject to} && K(x)u_l = f_l, \quad l = 1, \dots, m, \\
& && \rho^T x \leq M, \\
& && Ax \leq b, \\
& && x \in \{0, 1\}^n.
\end{aligned} \tag{22}$$

This program has m equilibrium equations, one per load condition, and the objective function considers m compliances. The relaxed master problem must include one cut per load condition. In other words, given a design vector x^k , an optimality cut $l_*^c(x, u_l^k, \nu_l^k) \leq y$ must be included in the master program for each load condition $l = 1, \dots, m$. In this case, the relaxed master problem becomes

$$\begin{aligned}
& \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\
& \text{subject to} && l_c^*(x, u_l^k, \nu_l^k) \leq y, \quad \begin{array}{l} k = 1, \dots, N, \\ l = 1, \dots, m, \end{array} \\
& && \rho^T x \leq M, \\
& && Ax \leq b, \\
& && x \in \{0, 1\}^n.
\end{aligned} \tag{P_{cm}^*}$$

Considering the representation of $l_c^*(x, u_l^k, \nu_l^k)$, given by (8), we can get an explicit form for the relaxed master problem for the multiple load minimum compliance problem (22). It is given by

$$\begin{aligned}
& \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\
& \text{subject to} && -\nu_l^{kT} u_l^k [x - x^k] - y \leq -f_l^T u_l^k, \quad \begin{array}{l} k = 1, \dots, N, \\ l = 1, \dots, m, \end{array} \\
& && \rho^T x \leq M, \\
& && Ax \leq b, \\
& && x \in \{0, 1\}^n.
\end{aligned} \tag{P_{cm}}$$

Important Remark 1. *All the results presented in [16], valid for the re-enforcement case, can be extended to the pure topology case (i.e., when $\eta_s = 0$ in (5)). Namely, Theorem 1, Proposition 2 (stating the validity of the l_c^* -cuts), and corollaries 1, 2, 3, in [16] can be extended with minor changes in the proofs (propositions 1 and 3 of [16] consider already $\eta_s \geq 0$). Therefore, the generalization of the presented GBD method applied to (2), (3) to the case $\eta_s \geq 0$ is straightforward.*

Important Remark 2. *The GBD method here presented, applied to problems (2), (3) can be extended to the case when multiple load conditions are imposed, including a feasibility cut $l_*^f(x, \lambda_l^k, \lambda_l^k, \phi^k) \geq 0$ whenever a considered load f_l , $l = 1, \dots, m$, leads to a set of non solvable equilibrium equations. If the equilibrium equations possess a displacement solution for a given load, a compliance cut $l_c^*(x, u_l^k, \zeta_l^k) \geq 0$ (in a minimum weight formulation, if $f_l^T u_l \geq C_l$, with C_l the compliance bound for the l th load case), or $l_c^*(x, u_l^k, \nu_l^k) \leq y$ (in a minimum compliance problem) is included.*

4 Statement of the Method

In this section, we present the formal statement of the Generalized Benders' Decomposition (GBD) method to solve the multiple load minimum weight problem (21). The assumptions considered are the same as in the previous sections, i.e. assumptions (A-1) – (A-4). The theoretical basis of the GBD algorithm for the minimum weight problem was presented in the first part of the article. The convergence proofs in [16] can be generalized to show finite convergence to a global minimum of the method stated below.

Algorithm 1: GBD for multiple load minimum weight problems

1. Set $P = Q = 1$, the upper bound $UB = +\infty$, the lower bound $y^* = -\infty$ and the convergence tolerance $\epsilon \geq 0$.
2. Solve the first relaxed master problem

$$\begin{aligned}
& \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\
& \text{subject to} && \rho^T x \leq y, \\
& && Ax \leq b, \\
& && x \in \{0, 1\}^n,
\end{aligned} \tag{M1}$$

by any solver for linear-mixed integer programming. If the problem is infeasible, stop and exit. If the problem is feasible, it necessarily has at least one optimal solution. This comes from the fact that $\rho^T x \geq 0, \forall x \in \{0, 1\}^n \implies y \geq 0$. Denote the solution of (M1) found by (x^*, y^*) , and its optimal value y^* .

3. Do for all load cases l : { If x^* is an infeasible design for the equilibrium equations $K(x^*)v = f_l$, compute a solution $\lambda^Q \in \mathbb{R}^d$ of the feasibility program

$$\begin{aligned}
H(x^*) = & \underset{\lambda \in \mathbb{R}^d}{\text{minimize}} && 0^T \lambda \\
& \text{subject to} && \lambda^T K(x^*) \lambda - f_l^T \lambda < 0, \\
& && f_l^T \lambda - C_l \geq 0.
\end{aligned}$$

Compute ϕ^Q and r_E^Q as

$$\begin{aligned}
\phi^Q &= ((K_1 \lambda^Q)^T \quad (K_2 \lambda^Q)^T \quad \dots \quad (K_n \lambda^Q)^T)^T, \\
r_E^Q &= -f_l^T \lambda^Q,
\end{aligned}$$

and set $Q \leftarrow Q + 1$.

Otherwise, if u^P satisfies the equilibrium $K(x^*)u^P = f_l$, compute the compliance $C_l^P = f_l^T u^P$. If $C_l^P > C_l$, compute ν^P and r_C^P as

$$\begin{aligned}
\nu^P &= ((K_1 u^P)^T \quad (K_2 u^P)^T \quad \dots \quad (K_n u^P)^T)^T \\
r_C^P &= C_l - f_l^T u^P - \nu^{PT} u^P x^*
\end{aligned}$$

and set $P \leftarrow P + 1$. Add other feasibility cuts if available (cf. section on combinatorial Benders' cuts). }

4. If x^* is a feasible design for all the equilibrium equations $K(x^*)u_l = f_l, l = 1, \dots, m$, and if $f_l^T u_l \leq C_l \forall l$ and $\rho^T x^* < UB$, then set $UB = \rho^T x^*$. If $UB - y^* \leq \epsilon$, then stop. The optimal design found is x^* , with optimal value $\rho^T x^*$. Otherwise, continue to step 5.

5. Solve the relaxed master problem

$$\begin{aligned}
& \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\
& \text{subject to} && \rho^T x \leq y, \\
& && -\nu^{kT} u^k x \leq r_C^k, \quad k = 1, \dots, P-1, \\
& && \phi^{kT} \lambda^k x \leq r_E^k, \quad k = 1, \dots, Q-1, \\
& && Ax \leq b, \\
& && x \in \{0, 1\}^n,
\end{aligned}$$

by any solver for linear-mixed 0 – 1 programming. Denote the solution of this program (x^*, y^*) , and its optimal value y^* . Return to step 3.

5 Accelerating GBD

Several ideas to accelerate the numerical performance of the Generalized Benders' Decomposition (GBD) are reported in the literature. An overview of different techniques used to accelerate GBD since this method was introduced in the early sixties is presented in [18]. We discuss in particular two of these techniques, since they are relevant for our investigation.

5.1 Pareto Optimal Cuts

In [14], it was proposed to discard cuts according to their relevance, measuring this by a Pareto dominance relationship. We apply this idea on the compliance cuts generated by GBD for the minimum compliance and minimum weight problems (2) and (3), respectively. The study presented in [14] was done on a min-max formulation problem, and therefore, for the problems we study here we need to adapt our definition of the Pareto dominance relationship.

Definition 1. A cut $l_*^c(x, u^j, \nu^j) \geq 0$ is said to dominate another cut $l_*^c(x, u^i, \nu^i) \geq 0$ if and only if

$$l_*^c(x, u^j, \nu^j) \geq l_*^c(x, u^i, \nu^i), \quad \forall x \in [0, 1]^n. \quad (23)$$

with a strict inequality for at least one design. A cut $l_c^*(x, u^j, \nu^j) \leq y$ is said to dominate another cut $l_c^*(x, u^i, \nu^i) \leq y$ if and only if

$$l_c^*(x, u^j, \nu^j) \leq l_c^*(x, u^i, \nu^i), \quad \forall x \in [0, 1]^n.$$

In [14], it was shown that for a general class of non-linear mixed-integer problems, the inclusion of non dominated Pareto optimal cuts can improve the rate of convergence of the lower bound towards the optimal solution. We have now the following result.

Proposition 3. Let $l_*^c(x, u^j, \nu^j) \geq 0$ and $l_*^c(x, u^i, \nu^i) \geq 0$ be two compliance cuts related to two designs $x^j, x^i \in [0, 1]^n$, with equilibrium solutions $u^j, u^i \in \mathbb{R}^d$ and Lagrange multipliers ν^j, ν^i , as (8), respectively. If $\eta_s = 0$ and $l_*^c(x, u^j, \nu^j) \geq 0$ dominates $l_*^c(x, u^i, \nu^i) \geq 0$, then

$$f^T u^j \leq f^T u^i.$$

The same condition applies for the cuts $l_c^*(x, u^j, \nu^j) \leq y$ and $l_c^*(x, u^i, \nu^i) \leq y$.

Proof. The relationship (23) is written explicitly as

$$C - f^T u^j + \nu^{jT} u^j (x - x^j) \geq C - f^T u^i + \nu^{iT} u^i (x - x^i), \quad \forall x \in [0, 1]^n. \quad (24)$$

If we evaluate the Pareto inequality (24) at $x = 0$, we obtain

$$-f^T u^j - \nu^{jT} u^j x^j \geq -f^T u^i - \nu^{iT} u^i x^i.$$

Using that $f^T u^j = \nu^{jT} u^j x^j + u^{jT} K_0 u^j$ we simplify this last condition to

$$2f^T u^j - u^{jT} K_0 u^j \leq 2f^T u^i - u^{iT} K_0 u^i. \quad (25)$$

In the case of pure topology problems ($\eta_s = 0$ in assumption (A-1)), $K_0 = 0^{d \times d}$, and (25) becomes

$$f^T u^j \leq f^T u^i.$$

The proof for the l_c^* -cuts is analogous. □

In other words, if $\eta_s = 0$, the Pareto dominance value of a cut $l_*^c(x, u^j, \nu^j) \geq 0$ (or a cut $l_c^*(x, u^j, \nu^j) \leq y$) is measured by the value of its compliance $c(x^j) = f^T u^j$. The lower the compliance, the higher is the corresponding Pareto dominance value.

This means that if we consider cuts related to a design with a low value in compliance, we expect these cuts to dominate in general a larger number of cuts than other cuts coming from designs with a higher value of compliance. We can therefore expect that these cuts are stronger and lead to faster convergence. This is computationally tested in the numerical examples section.

5.2 Combinatorial Benders' Cuts

When solving pure topology problems, we need to deal with the potential infeasibility of the equilibrium equations (1). For this purpose, GBD includes feasibility cuts, which are described in Section 3. However, the GBD feasibility cuts show in practice a weak capacity of improving the lower bound of the GBD algorithm. In order to improve the convergence of general linear mixed-integer optimization problems, [5] investigated the inclusion of the so called combinatorial Benders' cuts, which attack the mathematical source of the infeasibility of an inconsistent linear system. These cuts are the result of finding the set of subsystems of the original linear system, which are responsible for the infeasibility of the inconsistent linear system. This type of subsystem is called an "irreducible inconsistent subsystem of linear constraints" (IIS).

Given an inconsistent system, its set of IIS's is in general not unique, and the number of IIS's grows exponentially with the size of the problem (see [4]). The search of inconsistencies of a linear system has been studied by several authors, see for example [5, 9, 17, 24]. In the case of the problems (2) and (3), we need to find IIS's related to the equilibrium equations (1), when ever these equations are inconsistent for a given design. It is necessary to reformulate the equilibrium equations (1), to obtain an explicit relationship between an inconsistency of the equilibrium equations and the value of the design variables. The equilibrium equations (1) are rewritten as the system

$$\sum_{j=1}^n K_j z_j = f, \quad (26a)$$

$$z_j - u = 0, \quad \forall j \in \mathbb{N}_1(x), \quad (26b)$$

$$z_j = 0, \quad \forall j \in \mathbb{N}_0(x), \quad (26c)$$

where the index sets $\mathbb{N}_0(x)$ and $\mathbb{N}_1(x)$ are

$$\mathbb{N}_0(x) := \{j \in \{1, \dots, n\} \mid x_j = 0\},$$

$$\mathbb{N}_1(x) := \{j \in \{1, \dots, n\} \mid x_j = 1\}.$$

In the equations (26), we want to find out which, among the equations (26b) and (26c), are part of a short IIS (short IIS's discard a bigger set of designs from being feasible at the corresponding combinatorial cut). In order to find a minimal inconsistency set, the following result, based on Farkas's Lemma, is useful.

Theorem 2. *Given the inconsistent system $S = \{x \in Q^n \mid Cz = d\}$, the indices of the minimal infeasible subsystems of S are exactly the supports of the vertices of the polyhedron*

$$P = \{w \in Q^m \mid w^T C = 0, w^T d = -1\}.$$

Proof. The proof of this theorem is analogous to the one shown in [17], applied to an inconsistent system of linear inequalities $Cz \leq d$. \square

Theorem 2 is used to find IIS's related to the system (26) (i.e., when $Cz = d$ corresponds to the system (26)). There are several heuristics to generate IIS's (see, e.g. [5] and [9]). All of them are based on introducing an auxiliary Linear Program (LP) with feasible set in the polyhedron P . It is well known that an LP in a polyhedron P attains its extremes at the vertices of P . Wisely changing the coefficients of its objective function $f(y) = b^T y$ generates solutions at different vertices of the polyhedron P . For each solution y^* of the LP, and for each j_0 such that $y^*(j_0) \neq 0$, the equation numbered j_0 in (26) belongs to an IIS. Consequently, all non zero values of y^* indicates the indices of an IIS \mathbb{S}_l . For m' different objective vectors $b_1, \dots, b_{m'}$, m' different linear programs are formulated and solved, and m' solutions $y_1^*, \dots, y_{m'}^*$ can be found. These solutions generate at most m' IIS's $\mathbb{S}_1, \dots, \mathbb{S}_{m'}$. The shortest of them are taken to build combinatorial Benders' cuts (by shortest we mean, the IIS's containing the least number of indices). More specifically, we generate combinatorial Benders' cuts, using the following heuristic based on the Theorem 2.

Combinatorial Benders' cuts Algorithm

1. Consider a design x , such that the equilibrium equations (1) are inconsistent. Rewrite the equilibrium equations (26) as $Cz = d$. Set $l = 1$ and let b_l be the vector $\mathbf{1}$. Set the initial set of IIS's to $\mathbb{S}_l = \emptyset$, and set the maximum number of iterations to l_{\max} .
2. Solve the linear program

$$\begin{aligned} & \underset{y}{\text{minimize}} && b_l^T y \\ & \text{subject to} && C^T y = 0, \\ & && d^T y = -1. \end{aligned} \tag{27}$$

If this linear program is successfully solved to optimality and an optimal solution y^* is found, find the indices $I_0 = \{i \in \mathbb{N} : y^*(i) \neq 0\}$. I_0 is the set of indices numerating the constraints on the equilibrium equations (26). Then go to step 3. Otherwise, if the program (27) is infeasible or unbounded, set $\mathbb{S}_l = \emptyset$ and exit.

3. Form the set of indices \mathbb{S}_l in the following way.

Set $\mathbb{S}_l = \emptyset$. Do for all k : {

If $I_0(k)$ is related to one of the constraints $\sum_{j=1}^n K_j z_j = f$, do nothing. If $I_0(k)$ is related to one of the constraints $z_j - u = 0$ or $z_j = 0$, set $\mathbb{S}_l = \mathbb{S}_l \cup j$. }

4. Set $b_{l+1} = b_l$ and $l \leftarrow l + 1$.
5. Modify the objective coefficients $b_l(j)$ as: Do for all j {,

$$b_l(\mathbb{S}_l(j)) \leftarrow 2b_l(\mathbb{S}_l(j)).\}$$

In this way these indices become more expensive for the optimization program (27). If the maximum number of iterations has been reached, i.e. $l \geq l_{\max}$, go to step 6, otherwise, return to step 2.

6. Do for all l : { If $\mathbb{S}_l \neq \emptyset$, for each IIS \mathbb{S}_l , the combinatorial Benders' cut related to the inconsistent system (26) is given by

$$\sum_{j \in \mathbb{S}_l: x_j=0} x_j + \sum_{j \in \mathbb{S}_l: x_j=1} [1 - x_j] \geq 1. \}$$

Stop and exit.

5.3 GBD Heuristics for Finding Candidate Designs

In this section, we discuss some alternatives, based on different heuristics, to accelerate the performance of the GBD method applied to the minimum compliance and minimum weight problems, (22) and (21), respectively. As it was indicated in the first part of this section, the inclusion of high Pareto dominance cuts (related to low compliance designs) may accelerate the convergence of the GBD algorithm. We consider first, the application of the GBD algorithm to an alternative formulation of the structural design problem, in such a way that this modeling leads to a non convex projected problem. As it was indicated in the previous sections, the GBD algorithm applied to the minimum compliance (and minimum weight) problem converges to a global optimum in a finite number of steps. This result is valid, since the corresponding relaxed projected problem on the design space is convex. By dropping this convexity condition, the algorithm no longer guarantees the convergence to global optima. In exchange, we expect to observe a much faster convergence of the algorithm, and potentially good feasible designs could be generated in the process. One example of such modeling, dropping the convexity assumption, is given by the SIMP approach (see [2]), a material interpolation scheme with a penalization parameter $p > 1$. For instance, the SIMP interpolation scheme changes the original minimum weight problem (3), to the following mixed 0 – 1 problem

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, u \in \mathbb{R}^d}{\text{minimize}} && \rho^T x \\ & \text{subject to} && K_p(x)u = f, \\ & && f^T u \leq C, \\ & && Ax \leq b, \\ & && x \in \chi_D = \{0, 1\}^n, \end{aligned} \tag{28}$$

where the stiffness matrix $K(x)$ in the assumption (A-1) is replaced by the matrix $K_p(x)$ modeled by

$$K_p(x) = K_0 + \sum_{j=1}^n x_j^p K_j.$$

Clearly, formulation (28) does not change the value of the compliance, since $K_p(x) = K(x)$ for all $x \in \{0, 1\}^n$. However, the gradient of the compliance as a function of the design variables is changed by this modeling. Thus, after computing the Lagrange multipliers for the GBD method applied to the problem (28), we obtain the following form for the compliance cuts

$$C - f^T u^k + p \nu^{kT} u^k (x - x^k) \geq 0,$$

with ν^k given by (8). The GBD algorithm applied to any non convex model will, in this context, be called *GBD heuristics*, and when it is applied to a convex model, it will be called *global GBD*. If the GBD method is applied to this interpolation model for $p > 1$ (the compliance and its gradient are well defined in $\{0, 1\}^n$, and even in $[0, 1]^n$, for $p \geq 1$),

we expect convergence to occur quickly, but only few feasible designs of good quality (if any at all) are found. An alternative to overcome this difficulty is to smooth the modeling, by mixing a convex ($p = 1$) and a non convex ($p > 1$) interpolation scheme. Consider for example, the stiffness matrix

$$K_{\alpha_0, p}(x) = K_0 + \sum_{j=1}^n [\alpha_0 x_j + [1 - \alpha_0] x_j^p] K_j, \quad (29)$$

where a parameter $0 \leq \alpha_0 \leq 1$ is introduced to control the combination of these two different interpolation schemes. In practice, α_0 controls the speed of convergence of the algorithm. Using (29) to replace assumption (A-1), we obtain the mixed GBD compliance cuts for the minimum weight problem (3)

$$\begin{aligned} l_{\alpha_0}^c(x, u^k, \nu^k) &\geq 0, \\ l_{\alpha_0}^c(x, u^k, \nu^k) &= C - f^T u^k + \nu^{kT} u^k (D - D^k), \end{aligned} \quad (30)$$

while for the minimum compliance problem (2), the mixed GBD cuts are given by

$$\begin{aligned} l_c^{\alpha_0}(x, u^k, \nu^k) &\leq y, \\ l_c^{\alpha_0}(x, u^k, \nu^k) &= f^T u^k + \nu^{kT} u^k (D^k - D). \end{aligned} \quad (31)$$

The vectors $D, D^k \in \mathbb{R}^n$ are given by

$$\begin{aligned} D_j &= \alpha_0 + p[1 - \alpha_0]x_j, & j = 1, \dots, n, \\ D_j^k &= \alpha_0 + p[1 - \alpha_0]x_j^k, & j = 1, \dots, n. \end{aligned}$$

The value of p seems to play a secondary role, and for numerical examples, a value of $p = 2$ will be used. On the other hand, the introduced parameter α_0 connects homotopically the gradients of the compliance related to two interpolations schemes (in this case, SIMP with $p = 1$ and $p = 2$). This means that if the gradient of the compliance for $p = 1$ ($C_1(x)$) and $p = 2$ ($C_2(x)$) are respectively given by

$$\begin{aligned} \nabla_x C_1(x) &: [0, 1]^n \longrightarrow \mathbb{R}_-^n, \\ \nabla_x C_2(x) &: [0, 1]^n \longrightarrow \mathbb{R}_-^n, \end{aligned}$$

then, it exists a continuous function (a homotopy) $H : [0, 1]^n \times [0, 1] \longrightarrow \mathbb{R}_-^n$, given by

$$H(x, \alpha_0) = \alpha_0 \nabla_x C_1(x) + [1 - \alpha_0] \nabla_x C_2(x).$$

In particular, we have that

$$H(x, 0) = \nabla_x C_1(x), \text{ and } H(x, 1) = \nabla_x C_2(x).$$

Therefore, a convenient value for the mixture parameter α_0 will balance the the convergence of the GBD heuristics and its closeness to the global GBD algorithm. Another alternative for the modeling is to use the interpolation scheme suggested in [19]. By using this scheme we can expect similar results in convergence and capacity for generating potential solutions to the problem.

5.4 Numerical Procedure

When solving larger problems, the GBD method shows inefficiency in solving the relaxed master problem. This efficiency, measured in time consumed for solving the master problem to optimality, is in general unpredictable, as in general for combinatorial problems. Two ideas are proposed to overcome this serious problem. The first idea is based on the computational experience that in general, most of the time spent by mixed-integer solvers is used to prove optimality of the incumbent, while a small part of the time is spent in actually finding the optimal solution. Solving the master problem to optimality is important, because it allows us to measure precisely the lower bound, and therefore, to measure correctly the convergence of the algorithm at the current stage. Nevertheless, since the absolute gap between the lower and upper bounds is a non increasing sequence through the iterations, it is important to measure it precisely, only as the stopping criterion is about to be satisfied. We propose to solve the master problem to optimality only every fixed number of iterations. This idea should help economizing the computational time, that is generally wasted in proving optimality of master problems, when it is not important. The second idea is to use GBD heuristics, with the mixed interpolation scheme (29), to generate and accumulate many high Pareto value cuts, to be included as initial cuts for the GBD algorithm. This is expected to give a higher rate of convergence of the algorithm. The resulting algorithm, described qualitatively above, is stated properly now.

Algorithm 2: Method for the multiple load minimum weight problem (21)

1. Compute the solution x^c of the continuous relaxation of the minimum weight problem (21). Compute its compliances $c_1(x^c), \dots, c_m(x^c)$. Set $m_c = 0$. Do for all load cases $l = 1 \dots m$: { Set $f = f_l$. If $c_l(x^c) \geq C_l$, set $m_c \leftarrow m_c + 1$, compute the vectors $u_{m_c}^c, \nu_{m_c}^c$ and the corresponding compliance cut $C_c^{m_c} : l_*^c(x, u_{m_c}^c, \nu_{m_c}^c) \geq 0$, after (8). } Set $N_c = 0$.
2. Set $\alpha_0 = 0.01$. If $\eta_s = 0$, we set a temporary value (for instance $\eta_s = 0.001$). Set the stopping criteria for the master problem, for example a gap of 0.2% and a limit time of 100[s]. Set a time limit for the heuristic procedure (for instance $T_H = 3[h]$). Set $i = 1$.
3. Include the cuts $C_c^1, \dots, C_c^{m_c}$, and the cuts C^1, \dots, C^{N_c} (if $N_c > 0$) in the relaxed master problem (P_{wm}^*).
4. If $i > 1$, include the mixed cuts $C_{\alpha_0}^1, \dots, C_{\alpha_0}^{i-1}$ in the master problem. Solve the master problem (P_{wm}^*). Denote the incumbent of the master problem (x^*, y^*) .
5. Do for all load conditions $l = 1, \dots, m$: { Set $f = f_l$ and compute the compliance $c_l(x^*)$. If $c_l(x^*) \geq C_l$, then set $i \leftarrow i + 1$, compute the vectors u^i, ν^i after (8), and the mixed compliance cuts $C_{\alpha_0}^i : l_{\alpha_0}^c(x, u^i, \nu^i) \geq 0$ after (30). If $C_l \leq c_l(x^*) \leq L$, set $N_c \leftarrow N_c + 1$ and compute the cut $C^{N_c} : l_*^c(x, u^i, \nu^i) \geq 0$ after (8). }
6. If $c_l(x^*) \leq C_l$, for all $l = 1, \dots, m$, and $\rho^T x^* \leq UB$, set $UB = \rho^T x^*$.
7. Evaluate the stopping criterion $UB - y^* = 0$. If it is satisfied, then continue to step 8. Otherwise, return to step 4.
8. Increase the value of α_0 by 0.01. If the current time $t < T_H$, set $i = 1$ and return to step 3. Otherwise continue to step 9.

9. Set $\alpha_0 = 1.0$, and η_s to its original value, if it was changed in step 2. Set the stopping criteria for the master problem to a gap of 0.05% or a time limit of 24 [h] every 30 iterations, and a gap of 0.1% or a time limit of 300[s] otherwise.
10. Consider the cuts C^1, \dots, C^{N_c} and $C_c^1, \dots, C_c^{m_c}$ as initial cuts for the global GBD method. Run the GBD algorithm (Algorithm 1 in Section 4) until convergence, or until the computational time limit T_B is exceeded.

6 Implementation

The GBD method presented in this article is implemented in the numerical environment and high level programming language MATLAB ([15]), for solving 2-D truss topology optimization problems. The solver used in the treatment of the master problem is the commercial branch-and-cut solver for mixed-integer programming CPLEX version 9 ([12]). The linear programs that appear while generating combinatorial Benders' cuts are solved by the Simplex solvers in CPLEX. The continuous relaxations of the minimum compliance and minimum weight problems are modeled using the SIMP interpolation scheme [2], and are numerically solved using the Method of Moving Asymptotes (MMA), see [21, 23]. The MATLAB solver for non-linear optimization `fmincon` was used to compute optimal solutions of the feasibility problem (17) and the feasibility problem used to compute the SDP cuts (see the first part of the article [16]).

7 Numerical Examples for Truss Topology Optimization Problems

In this section, we present the numerical experience with the Generalized Benders' Decomposition (GBD) method applied to a set of structural topology optimization problems. Specifically, we attack truss topology optimization examples of the minimum weight and minimum compliance problems, (21) and (22), respectively. A truss structure is an assembly of long slender bars connected at frictionless nodes. The external loads are applied only at the nodes. The design variables of the problems represent the area of the potential bars in the structure. These areas belong to the discrete set $x_j \in \{0, t_1, \dots, t_\ell\}$, where $0 < t_1 < \dots < t_\ell < +\infty$ are given values. The Young modulus E_i is scaled to unity for all potential bars. The same is valid for the mass density. Therefore, we use the terms weight, mass, and volume as equivalent ones. The tolerance for the feasibility of all constraints is set to 10^{-5} . All examples were run on an UltraSPARC IV processor, running at 1800 MHz.

7.1 Pareto cuts I

We start by investigating the numerical influence on the convergence speed for the Benders' algorithm (Algorithm 1 in Section 4), when high Pareto value cuts are included, i.e., compliance cuts obtained from designs with a low objective value. We consider the program

$$\begin{aligned}
 & \underset{x \in \mathbb{R}^n}{\text{minimize}} && f^T K(x)^{-1} f \\
 & \text{subject to} && \rho^T x \leq M, \\
 & && Ax \leq b, \\
 & && x \in [\epsilon, 1]^n,
 \end{aligned} \tag{32}$$

which is the (slightly perturbed) natural continuous relaxation of the nested formulation of the minimum compliance problem in a single load case (see [2]). Here, ϵ represents

Objective	C_c	Objective	CPU [h:m:s]	Iter.	Gap (%)
Compliance	yes	2.8978	0:17:44	233	0.030
single load	no	2.8978	0:42:36	460	0.495
Compliance	yes	3.0919	0:00:21	21	0.368
multiple load	no	3.0919	0:00:47	67	0.029
Weight	yes	9.3896	1:51:36	456	0.00
single load	no	20.4570	72:03:25	3333	84.323
Weight	yes	10.4787	01:23	79	0.00
multiple load	no	10.4787	32:34	498	0.00

Table 1: Comparison of the performance of GBD including/discarding the compliance cut(s) C_c^l , $l \in J$, from the solution of the continuous relaxation.

a very small positive number. From the solution of this problem, we obtain an almost optimal Pareto cut. This special cut is used for testing its influence on the convergence speed on the Benders' algorithm. Since (32) is a convex problem when the stiffness matrix is affine in the design variables (for a proof, see [22]), any local minimum x^c is also a global solution. x^c is in general, a non 0 – 1 design. This means that the compliance cut $C_c := l_c^*(x, u(x^c), \nu(x^c)) \geq 0$ is a valid inequality in $[0, 1]^n$, and in particular, in $\{0, 1\}^n$. For the minimum weight problem (2) and in the multiple load case, the cuts C_c^l , $l \in J$ (where J is an appropriate index set) are constructed analogously, by using the continuous relaxation of the corresponding problem.

Any global solution x^c of (32) has the property that its corresponding compliance cut C_c is not dominated by any other feasible compliance cut in $[\epsilon, 1]^n$. Therefore, C_c is an optimal Pareto cut, with respect to the Pareto dominance in Definition 1. Our purpose is to test the influence of C_c on the convergence of the Benders' algorithm. x^c is calculated by the solver for nonlinear optimization MMA (see [21]). We consider a cantilever example. The design domain is shown in Figure 1(a). The 31 bar ground structure for this example appears in Figure 1(b). For this example, we used a re-enforced structure with a stiffness matrix given by (4) and (5), with $\eta_s = 0.01$, and a compliance bound of $C = 3.0$, or mass bound of $M = 9.5$. Four instances of the problem were considered: Two minimum weight, and two minimum compliance problems. Each of the four instances was solved twice, using the standard Benders' algorithm (Algorithm 1 for multiple load minimum weight problems, and its corresponding version for the minimum compliance problem). One instance included the cut(s) C_c^l , $l \in J$ from the beginning of the algorithm, while the others did not. A general comparison of the performance of the algorithm on these four examples is shown in Table 1. In this table, we have included the value of the objective function of the best found feasible design, the CPU time for the Benders' algorithm, the number of iterations (master problems solved), and the relative optimality gap. The relative optimality gap is defined as

$$\text{Gap} := \frac{UB - LB}{UB} \times 100,$$

with UB = upper bound for the global optimum (the objective function value of the incumbent), LB = lower bound for the global optimum in the current iteration. The stopping criterion of the algorithm is set for this example to $\text{Gap} \leq 0.5\%$. The optimal solution to the single load minimum compliance example is shown in Figure 1(c).

Table 1 shows that in seven out of the eight instances, the stopping criterion was satisfied within the time limit of 72[h]. We note an important difference in the CPU time

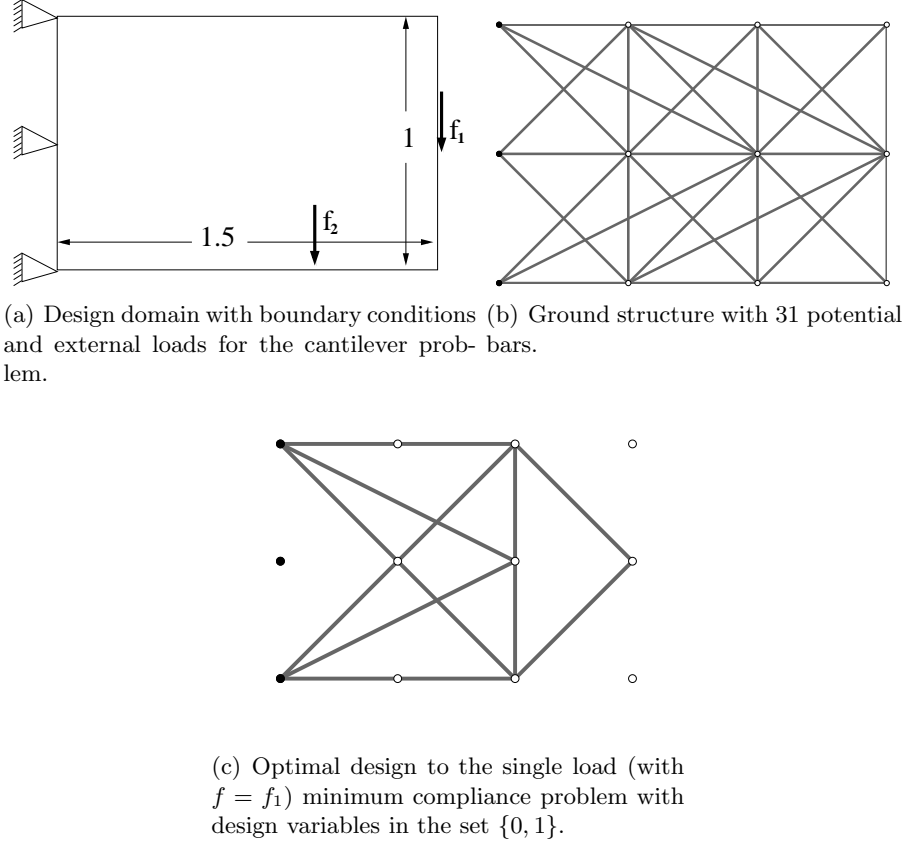


Figure 1: The single load cantilever example.

and the number of iterations executed. In the cases where the optimal Pareto cut C_c was included, both the CPU time, and the number of iterations were considerably lower. The influence was in fact bigger on the minimum weight problems. In general, these differences in performance show a clear positive influence of including the special cut(s) C_c^l from the solution of the continuous relaxation of the problems. The strength of the cut(s) C_c^l , after the Pareto dominance relationship (23), relies on the low value of the compliances of x^c . Consequently, we suggest that cuts related to low compliance designs should have a larger influence on the convergence of the Benders' algorithm. This implies that if we can find low compliance designs by any mean, and include their corresponding compliance cuts at the beginning of the algorithm, we can expect an acceleration of the convergence of the algorithm. One example of a way to find low compliance designs is considering solutions coming from the SIMP interpolation scheme for the problems (2) and (3) (or (22) and (21)), with high penalization (say $p > 3$), in order to achieve convergence to (almost) 0 – 1 designs (see [2]). We will come back to this point after examining the influence of combinatorial Benders' cuts for pure topology design problems.

7.2 Combinatorial Benders' cuts and SDP cuts

We investigate the use of combinatorial Benders' and SDP feasibility cuts, in order to accelerate the performance of the GBD method. These techniques apply only to pure topology optimization problems, where the assumption (A-1) is taken with $\eta_s = 0$. Combinatorial Benders' cuts were briefly described in Section 5, and SDP cuts were introduced in the first part of this article [16].

The inclusion of combinatorial Benders' and SDP cuts for infeasibility of the equi-

librium equations (1) is tested on the same single load minimum compliance cantilever example as shown above, but in the pure topology case ($\eta_s = 0$). We include two initial compliance cuts, one generated by the solution from the continuous relaxation x^c , and the other cut, C_S , generated by the solution for the SIMP interpolation scheme, x_S , with a final penalization of $p = 6$. In Table 2, we can see the results in performance comparing the cases including/not including combinatorial Benders' cuts and/or SDP cuts. The table shows the objective function value upon termination, the CPU time consumed (CPU), the number of master problems solved (Iter.), the number of Benders' feasibility cuts (Feas. cuts), combinatorial Benders' cuts (Comb. cuts), and SDP cuts (SDP cuts). The last column states the final relative convergence gap (Gap). The termination gap was set to 4%. For this problem, the number of master problems solved and the computation time decreases as combinatorial Benders' cuts and SDP cuts are included. This result indicates a numerical advantage in including combinatorial Benders' cuts and SDP cuts in the master problem at each iteration, when treating pure topology problems. Note that there is an important difference in the convergence of the algorithm when solving the pure topology optimization problem, with respect to the re-enforced version of the same problem. While the example considering re-enforcement ($\eta_s = 0.01$) converges in reasonable time, the pure topology example ($\eta_s = 0.0$) takes a much longer time (34[h]:24[m] compared to 17[m]:44[s] of the re-enforcement example) and many more iterations (1101 iterations, compared to 233 for the re-enforcement case) to achieve a much higher final gap (Gap = 3.948% compared to 0.03%). This result suggests that the Benders' algorithm might be better suited for re-enforcement problems than pure topology optimization problems.

Objective	CPU[h:m]	Iter.	Feas. cuts	Comb. cuts	SDP cuts	Gap (%)
2.946	46:34	1370	668	0	0	3.948
2.946	33:50	1169	437	446	0	3.948
2.946	34:24	1101	412	216	412	3.948

Table 2: Comparison of the performance of GBD including/not including combinatorial Benders' cuts and/or SDP feasibility cuts.

7.3 Pareto cuts II

A third experience is made, considering the same cantilever example, but including 5 potential areas for each bar, namely, $x_j \in \{0, 0.2, 0.4, 0.6, 0.8, 1.0\}$. This modification increments by 5 times the number of design variables in the problem. We set the stopping criterion to a relative gap of 0.5%. We include the compliance cut related to the solution x^c of the relaxation (32) (C1). The results in Table 3 show that the CPU time limit of 72[h] was reached, and the algorithm stopped before converging. Nevertheless, in this example, only 14 iterations were executed. The algorithm spent 5[m]:32[s] in iterations 1 – 12, 43[h]:20[m]:24[s] in iteration 13, and 43[h]:05[m]:05[s] for iteration 14.

This example shows how in some situations, the solution of two consecutive master problems can have such a different CPU time, which may have a huge impact on the performance of the algorithm. We propose two actions to avoid this situation. First, we introduce two relative tolerances for the resolution of the master problem. One of the gaps is small (for example 0.05%), and it is used only every a certain number of instances of the master problem. The other tolerance, the bigger one (for example 0.2%), is used in most of the master problems. The second idea is to impose two time limits for execution

Case	UB	LB	CPU [h:m:s]	Iter.	Pareto cuts	Gap (%)
C1	2.873	2.657	86:31:26	14	0	7.52
C2	2.706	2.698	0:12:02	31	327	0.30

Table 3: Comparison of the performance of GBD for a multi-area optimization example, including only the compliance cut C_c related to the solution of the continuous relaxation (C1), and the use of Algorithm 2 (C2).

of each master problem, depending on the corresponding tolerance of each iteration. This implies that we could eventually exit a master problem with only an incumbent, which is not optimal with respect to the given optimality tolerance. At the same time, we want to use mixed Benders’ compliance cuts to generate as many initial cuts for the Benders’ algorithm as possible. Considering these two points, a modified algorithm was introduced (Algorithm 2, described in Section 5). We used this algorithm to solve this problem, and name this experience C2. This algorithm includes Benders’ mixed cuts (given by (30) and (31)) to find many candidates designs with a high Pareto dominance value, to be included in the master problem.

The experiment C2 reached the stopping criterion after 31 iterations in 12[m]:02[s] after a heuristic stage time of 7[m]:44[s], generating 327 compliance cuts, see Table 3. In these examples, no feasibility cuts were generated. A significant improvement due to the inclusion of the heuristic stage of Algorithm 2 is observed. Not only an improvement in the relative gap is attained, but also the computational time is dramatically reduced.

7.4 Numerical Experience

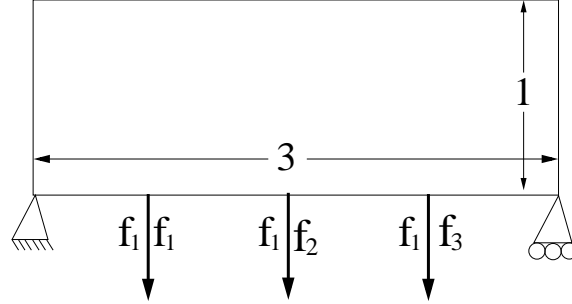
We attack 12 problem instances of the minimum compliance (22) and minimum weight (21) problems, for a particular geometry of the design domain. Both single and multiple load conditions, and single and multiple bar areas are considered. We also attack both, the re-enforcement ($\eta_s > 0$) and the pure topology ($\eta_s = 0$) instances of the problems.

The considered geometry is presented in Figure 2(a). The design domain is discretized into a 74 bar ground structure, with 28 degrees of freedom, see Figure 2(b). Table 4 presents the basic description of the examples P1–P12, including the name of the problem, the number of areas available for each bar (Areas), the number of total design variables of the problem (DV), the value of the re-enforcement parameter η_s , the objective function, the number of load cases considered, the magnitude of the external load(s), and the weight or compliance bound.

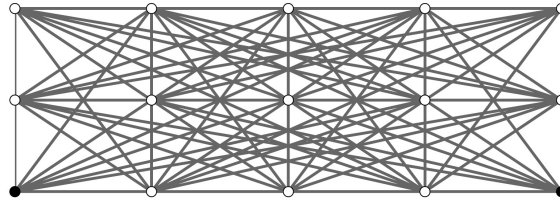
In examples P1–P6, we solve the minimum compliance problem (22), while in examples P7–P12 we solve the minimum weight problem (21). In these examples we consider two sets of available areas: The single area ($x \in \{0, 1\}$) case instances P1, P2, P7, and P8; and a multiple set of areas ($x \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$) in examples P3–P6 and P9–P12. A single load condition is considered in problems P1–P4, and P7–P10. The load condition is given by $f = f_1 + f_2 + f_3$ as shown in Figure 2(a). Similarly, multiple load conditions are considered in problems P5–P6 and P11–P12. The multiple load conditions consist of the loads f_1 , f_2 , and f_3 as independent load scenarios. Moreover, P1, P3, P5, P7, P9 and P11 are pure topology optimization problems ($\eta_s = 0.0$), while P2, P4, P6, P8, P10 and P12 are re-enforcement instances ($\eta_s = 0.01$).

P	Areas	DV	η_s	Objective	Load cases	Load	Weight $\rho^T x$	Comp. $\max_{1 \leq l \leq m} \{f_l^T u_l\}$
P1	1	74	0.0	Compliance	1	1.0	23.5	opt
P2	1	74	0.01	Compliance	1	1.0	23.5	opt
P3	5	370	0.0	Compliance	1	1.0	16.0	opt
P4	5	370	0.01	Compliance	1	1.0	16.0	opt
P5	5	370	0.0	Compliance	3	10.0	23.5	opt
P6	5	370	0.01	Compliance	3	10.0	23.5	opt
P7	1	74	0.0	Weight	1	1.0	opt	0.0895
P8	1	74	0.01	Weight	1	1.0	opt	0.8773
P9	5	370	0.0	Weight	1	1.0	opt	0.1134
P10	5	370	0.01	Weight	1	1.0	opt	0.1107
P11	5	370	0.0	Weight	3	10.0	opt	1.5638
P12	5	370	0.01	Weight	3	10.0	opt	1.5325

Table 4: Problem statistics for the 12 problem instances.



(a) Design domain with boundary and load conditions for the bridge like example.



(b) Ground structure with 74 bars.

Figure 2: Design domain and ground structure considered in the numerical examples.

7.5 Results

Algorithm 2 from Section 5, modifying the standard Benders' algorithm, was used for attacking all the problem instances in Table 4. The heuristic stage of the algorithm (Step 1 - 9) was limited to a maximum duration of $T_H = 3[h]$, while the global Benders' part of the algorithm was limited to $T_B = 72[h]$. The results are shown in Tables 5 and 6. In these tables, we show the objective function value of the best feasible design found, the number of iterations of the global Benders' stage of the algorithm (Benders' iter.), the final relative optimality gap (Gap), the number of Pareto candidates generated during the heuristic stage of the algorithm (Pareto cuts), and the CPU time for the heuristics (CPU 1) and the global Benders' (CPU 2) stages of the algorithm. The optimal designs for three (P2, P5, and P11) of the twelve examples are shown in Figures 3, 4, and 5. In

the examples with multiple areas, the figures show integer numbers besides the bars to indicate the relative comparison among the areas.

P	$\max_l \{C_l\}$	Benders' iter.	Gap (%)	Pareto Cuts	Feas. Cuts	CPU 1 [h:m]	CPU 2 [h:m]
P1	0.0895	1140	0.98	85	0	03:02	72:02
P2	0.0877	60	0.44	115	0	03:03	00:11
P3	0.1134	145	0.44	594	0	03:19	04:26
P4	0.1107	83	0.49	652	0	03:10	02:46
P5	1.5638	90	0.36	222	0	03:15	79:22
P6	1.5325	30	0.39	241	0	03:15	04:34

Table 5: Statistic obtained for examples P1 – P6.

P	Weight	Benders' iter.	Gap (%)	Pareto Cuts	Feas. Cuts	CPU 1 [h:m]	CPU 2 [h:m]
P7	23.430	1091	1.46	129	0	03:06	72:01
P8	23.930	6	0.22	145	0	03:02	00:02
P9	16.084	450	1.17/0.83*	648	3	03:15	94:00
P10	16.085	510	1.18/0.86*	585	0	03:09	74:52
P11	23.486	120	0.24	279	0	03:03	19:22
P12	23.504	180	0.37	241	0	03:19	25:56

Table 6: Statistic obtained for examples P7 – P12. * The relative optimality gap based on the lower bound obtained from solving the continuous relaxation of the considered problem.

For eight of the examples the stopping criterion ($\text{Gap} \leq 0.5\%$) was reached within the given time limits. The other four examples reached a final relative optimality gap $< 1.5\%$. The number of global Benders' iterations executed varies from 6 to 1140. Example P9 is the only one for which feasibility cuts were generated (P9 generated one SDP cut, one Benders' feasibility cut, and one combinatorial Benders' cut). This means that, in general, the incumbents of the master problems correspond to feasible designs for the equilibrium equations. This could be interpreted in the following way. If sufficiently many good compliance cuts are introduced during the heuristic stage of the algorithm, a sufficiently good approximation of the compliance function is made (at least in the neighborhood of the global solutions that we are trying to find). This could prevent designs, which are infeasible for the equilibrium equations (1), to be even close to be optimal in the relaxed master problems, as it really is the case on the original mixed 0 – 1 problems. This numerical experience suggests that the inclusion of many good candidate designs in the final stage of the Benders' algorithm, not only improves the rate of convergence, but also helps to prevent stepping into equilibrium infeasible designs.

8 Final Remarks and Future Work

The Generalized Benders' Decomposition (GBD) method applied to minimum compliance and minimum weight truss topology optimization problems has shown to be able to solve medium size problem instances to global optima. Several techniques, such as the combinatorial Benders' cuts and heuristic variations of the Benders' algorithm, have been

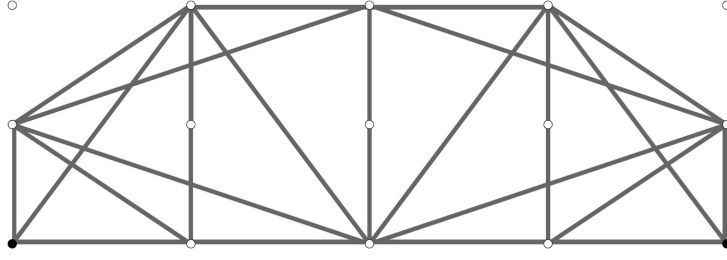


Figure 3: Optimal design for example P2.

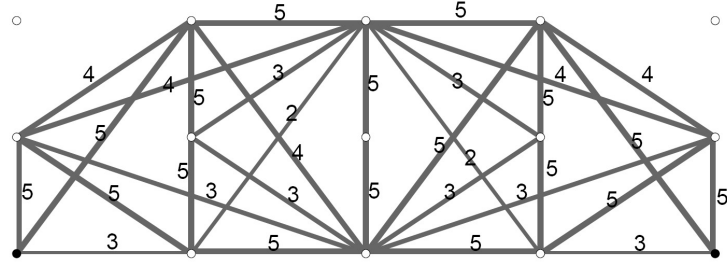


Figure 4: Optimal design for example P5.

successfully applied to accelerate the method in the numerical examples. One important future aspect is to extend the GBD method to structural topology optimization problems with local stress and/or displacement constraints. The inclusion of this kind of local failure criteria is an important extension of the algorithm, especially for industrial engineering applications. In order to achieve this, an investigation about the mathematical properties of the different failure criteria suggested in the literature must be done. In particular convexity is an important issue, in order to guarantee global optimality of a GBD approach. We also propose to investigate other classes of optimal design problems, that could be well suited for GBD. For example, we will consider more advanced modeling situations and finite elements, such as plate and shell elements, and optimal designs of composite structures. We will also investigate the possibility to solve other topology optimization problems, such as the design of compliant mechanisms and maximum stiffness problems with stability constraints by GBD.

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References

- [1] W. Aichtziger and M. Kočvara. Structural topology optimization with eigenvalues. *SIAM Journal on Optimization*, 18(4):1129–1164, 2007.
- [2] M.P. Bendsøe and O. Sigmund. *Topology Optimization: Theory, Methods and Applications*. Springer-Verlag, 2nd edition, 2003.

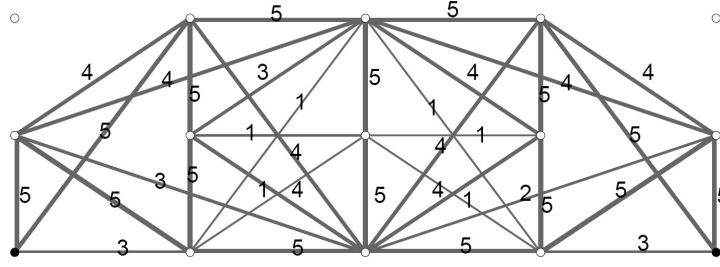


Figure 5: Optimal design for example P11.

- [3] S.C. Brenner and L.R. Scott. *The mathematical theory of finite element methods*. Springer, 1994.
- [4] N. Chakravarti. Some results concerning post-infeasibility analysis. *European Journal of Operational Research*, (73):139–143, 1994.
- [5] G. Codato and M. Fischetti. Combinatorial Benders’ cuts for mixed-integer linear programming. *Operations Research*, 54(4):756–766, 2006.
- [6] B.N. Datta. *Numerical Linear Algebra and Applications*. Brooks/Cole Publishing, 1995.
- [7] P. Deufhard and A. Hohmann. *Numerical Analysis in Modern Scientific Computing*. Springer, 2003.
- [8] A.M. Geoffrion. Duality in non linear programming: a simplified applications-oriented development. *SIAM Review*, 13(1):1–37, 1971.
- [9] J. Gleeson and J. Ryan. Identifying minimal infeasible subsystems of inequalities. *ORSA Journal on Computing*, 2(1):61–63, 1990.
- [10] O.K. Gupta and A. Ravindran. Branch and bound experiments in convex nonlinear integer programming. *Management Science*, 31(12):1533–1546, 1985.
- [11] J. Hiriart-Urruty and C. Lemaréchal. *Convex Analysis and Minimization Algorithms II*. Springer-Verlag, 1996.
- [12] ILOG Inc. *CPLEX 9.0 Users Manual*, 2003.
- [13] A.H. Land and A.G. Doig. An automatic method of solving discrete programming problems. *Econometrica: Journal of the Econometric Society*, 28(3):497–520, 1960.
- [14] T.L. Magnanti and R.T. Wong. Accelerating Benders decomposition: algorithmic enhancement and model selection criteria. *Operations Research*, (29):464–484, 1981.
- [15] The MathWorks, Inc. *Matlab user manual version 7.1*, 2005.
- [16] E. Munoz. *Global Optimization for Discrete Topology Design problems By Generalized Benders’ Decomposition*. PhD thesis, Department of Mathematics, Technical University of Denmark, 2010. Chapter 3.
- [17] M. Parker and J. Ryan. Finding the minimum weight IIS cover of an infeasible system of linear inequalities. *Annals of Mathematics and Artificial Intelligence*, 17(1-2):107–126, 1996.

- [18] W. Rei, J. Cordeau, M. Gendreau, and P. Soriano. Accelerating Benders decomposition by local branching. *INFORMS Journal on Computing*, 21(2):333–346, 2009.
- [19] M. Stolpe and K. Svanberg. An alternative interpolation scheme for minimum compliance topology optimization. *Structural and Multidisciplinary Optimization*, (22):116–124, 2001.
- [20] R.A. Stubbs and S. Mehrotra. Generating convex polynomial inequalities for mixed 0–1 programs. *Journal of Global Optimization*, 24(3):311–332, 2002.
- [21] K. Svanberg. Method of moving asymptotes - a new method for structural optimization. *International Journal for Numerical Methods in Engineering*, 24(2):359–373, 1987.
- [22] K. Svanberg. On the convexity and concavity of compliances. *Structural and Multidisciplinary Optimization*, 7(1-2):42–46, 1994.
- [23] K. Svanberg. A class of globally convergent optimization methods based on conservative convex separable approximations. *SIAM Journal on Optimization*, 12(2):555–573, 2002.
- [24] J.N.M. van Loon. Irreducibly inconsistent systems of linear inequalities. *European Journal of Operational Research*, 8(3):283–288, 1981.
- [25] L.A. Wolsey and G.L. Nemhauser. *Integer and combinatorial optimization*. Wiley-Interscience, 1998.

Article 3

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A New Generalized Benders' Decomposition Method for Topology Optimization Using Level Set Cuts

Eduardo Munoz*

June 6, 2010

Abstract

This article considers the use of a new type of Generalized Benders' Decomposition (GBD) method, modifying the classical procedure. GBD solves non linear mixed integer problems, by solving a sequence of linear mixed integer problems. The classical method includes at each instance of this sequence, one or several linear constraints (or cuts), obtained from the solution of the previous problem in the sequence. The new proposed method considers the level set of an upper bound for the considered objective function. Then, it searches for a non (necessarily) feasible point at this level set, and forms a GBD cut from this point. This new type of cuts are stronger than the classical GBD cuts and numerical results show that in practice they lead to faster convergence. The method is derived theoretically and specifically for classical structural topology problems. However, it could be generalized to a larger class of non-linear mixed integer problems, where the mixed problem can be reformulated as an integer problem with a continuous convex relaxation. In this case, global solutions are guaranteed. A set of numerical benchmark examples for structural topology optimization problems are solved to global optimality.

Mathematical Subject Classification (2000): 90C90, 74P05, 74P15

Keywords: Structural Topology Optimization, Global Optimization, Generalized Benders' Decomposition.

1 Introduction

We present a new type of Generalized Benders' Decomposition (GBD) approach ([4, 6]), which modifies the original nature of the technique. As its name shows, GBD is derived from the Benders' Decomposition (BD) method, which is a well known technique for solving mixed integer optimization problems, published in 1962 by Benders (see [1]). BD was originally designed for solving linear mixed integer problems. Geoffrion ([4]) and Lazimy ([6]) generalized the algorithm to a large class of non linear mixed integer problems, providing the name GBD. Several authors have indicated a number of techniques to accelerate the performance of the GBD algorithm (see for example [8], [16], [12], and [10]). The new method has been developed in a framework of structural topology problems. Previously, a GBD method with a heuristic preprocess was developed for topology problems in ([11] and [10]). The method presented there showed robustness in finding global solutions within tight optimality tolerances. In particular, the strength of the method depends on the quality of the intermediate found points, measured by a Pareto dominance relationship

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(see [8] and [16]). This feature we want to exploit in a special way, and we do this, by introducing some changes to the classical method.

The classical GBD method replaces a non linear mixed integer problem by a sequence of linear mixed integer optimization programs (called relaxed master problems, or simply master problems). The solutions of the master problems, under special conditions, converge to a solution of the non linear mixed integer problem. The method uses the following principle. After solving each linear mixed problem, a linear constraint (a cut) is added to the master problem, which improves the approximation of the original problem. The classical method includes cuts, which approximate the feasible set or the projected objective function value at the solution of the current master problem. The new proposed method follows as well the scheme of a sequence of master problems including at each iteration, one or several cuts making an approximation of the non linear problem at a given point. The difference is that the point of approximation does not need to be integer, and it is chosen at the level set of the best available upper bound for the non linear functions included in the problem. This implies that the sequence of linear problems approximates the non linear functions at points in a given level set. When enough cuts are included, and the approximation of to the level set is good enough, all designs with worse objective value are excluded. At this point, the algorithm finds a better solution if it exists, or converges finding an optimal solution. We find these points at the level set value by a simple convex combination between the integer solutions of the master problems, and the solution of the continuous relaxation of the problem.

This article is organized as follows: Section 2 presents the mixed 0–1 formulation of the minimum compliance problem, and the assumptions considered throughout this article. Section 3 and 5 describe the theoretical results supporting this new GBD method. Section 6 states formally the proposed method. A proof for the convergence of the method to a global optimum in a finite number of steps is presented in Section 7. Section 8 describes briefly the numerical implementation of the method used in the numerical tests. Section 9 presents the numerical results obtained with this method. Finally, Section 10 discusses briefly the obtained results, and suggests the future directions that should follow this work.

2 Problem Statement

The problem setting and assumptions are almost identical to the ones stated in ([11] and [10]). We consider a closed-bounded design domain $\Omega_c \subset \mathbb{R}^2$ or $\Omega_c \subset \mathbb{R}^3$, with piece-wise differentiable boundary. After a finite element discretization process, the design space consists in a set of design elements, where one or several 0-1 variables are linked to each of the design elements. The design vector (or design variable) $x \in \{0, 1\}^n$ represents a point in the design space. n is the number of design variables included in the problem. The design problem we study consists in finding the optimal vector $x^* \in \{0, 1\}^n$ minimizing an objective function. In the context of Structural Optimization, any choice of a vector $x \in \{0, 1\}^n$ represents a particular instance of the design of a structure. The design problem considers as well a set of given static external forces $f_l \in \mathbb{R}^d$, $l = 1, \dots, m$. The response of the structure to these loads is computed by the finite element method, which in the case of linear elasticity, results in the equilibrium equations

$$K(x)u_l = f_l, \quad l = 1, \dots, m, \quad (1)$$

where the matrix $K(x) \in \mathbb{R}^{d \times d}$ is called the stiffness matrix, and it is a function of the design variable x . The solutions $u_l \in \mathbb{R}^d$, $l = 1, \dots, m$ of (1) are the displacements of the structure under each load condition. d represents the number of degrees of freedom

in the structure induced by the discretization, and m is the number of external static load conditions. The design problem studied in this article is the minimum compliance problem, formulated as

$$\begin{aligned}
& \underset{\substack{x \in \mathbb{R}^n \\ u_1, \dots, u_m \in \mathbb{R}^d}}{\text{minimize}} && \max_{1 \leq l \leq m} \{f_l^T u_l\} \\
& \text{subject to} && K(x)u_l = f_l, \quad l = 1, \dots, m, \\
& && \rho^T x \leq M, \\
& && Ax \leq b, \\
& && x \in \{0, 1\}^n,
\end{aligned} \tag{2}$$

where $M > 0$ is the mass or weight limit for the design, and $\rho \in \mathbb{R}^n$ is the vector of material densities for each design variable. This problem is a non-linear mixed 0 – 1 program.

The projection theorem can prove that if $(x^*, u^*) \in \{0, 1\}^n \times \mathbb{R}^d$ is a solution of the problem (2), then x^* is a solution of the following projected problem.

$$\begin{aligned}
& \underset{x \in \mathbb{R}^n}{\text{minimize}} && \max_{1 \leq l \leq m} \{c_l(x)\} \\
& \text{subject to} && \rho^T x \leq M, \\
& && Ax \leq b, \\
& && x \in \{0, 1\}^n,
\end{aligned} \tag{3}$$

where the functions $c_l : [0, 1]^n \longrightarrow \bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$, $l = 1, \dots, m$, are defined as

$$c_l(x) = \begin{cases} f_l^T u_l(x), & \text{if } u_l(x), \text{ solution of } K(x)u_l(x) = f_l, \text{ exists.} \\ +\infty, & \text{otherwise.} \end{cases} \tag{4}$$

Throughout the article, we will call x^* any solution of problem (3), and denote C^* the optimal compliance of (3).

2.1 Assumptions

The assumptions considered are similar to the assumptions in [10], but generalized to multiple load conditions.

- (A-1) The stiffness matrix $K(x)$ is symmetric, affine in x , and positive semi-definite for all $x \in \{0, 1\}^n$. The matrix $K(x)$ is given by

$$K(x) = K_0 + \sum_{j=1}^n x_j K_j, \tag{5}$$

where $K_j \in \mathbb{R}^{d \times d}$ is the symmetric positive semi definite element stiffness matrix for the j -th design variable, and $K_0 \in \mathbb{R}^{d \times d}$ is a given symmetric positive semidefinite matrix ($K_0 \succeq 0$). We assume that

$$K_0 = \eta_s \sum_{j=1}^n K_j, \tag{6}$$

with $0 \leq \eta_s \ll 1$. In re-enforcement problems, we have $\eta_s > 0$, while $\eta_s = 0$ in pure topology problems. η_s is the re-enforcement parameter of the problem.

- (A-2) The mass limit, M satisfies $0 < M < \sum_{j=1}^n \rho_j$, where $\rho_j \geq 0$ for $j = 1, \dots, n$.

- (A-3) Each of the loads f_1, \dots, f_m is non null, i.e., $f_l \in \mathbb{R}^d \setminus \{0\}, \forall l = 1, \dots, m$.

- (A-4) The feasible set related to the constraints $Ax \leq b$ is non empty.

3 Generalized Benders' Decomposition Framework

In [11] and [10], a formulation of the Generalized Benders' Decomposition (GBD) method applied to minimum compliance and minimum weight problems was studied from theoretical and numerical points of view. These two articles follow the scheme of [6] for treating the subproblem (i.e., the problem obtained by fixing the integer variables) of the GBD method. Nevertheless, in [11] it was proven that the method defined by ([6]) is not available for problem (3), and a different subproblem was proposed and proved to be suitable.

From [10], it can be implicitly inferred, by a generalization process, that the minimum compliance problem (3) can be replaced by the following GBD master problem

$$\begin{aligned}
& \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\
& \text{subject to} && l_c^*(x, u_l^k, \nu_l^k) \leq y, & k &= 1, \dots, p, \\
& && l_c^*(x, u_l^k, \nu_l^k) \leq y, & l &= 1, \dots, m, \\
& && l_*^f(x, \lambda_l^k, \phi_l^k) \geq 0, & k &= 1, \dots, q, \\
& && \rho^T x \leq M, \\
& && Ax \leq b, \\
& && x \in \{0, 1\}^n.
\end{aligned} \tag{7}$$

In Problem (7), the function l_c^* was defined as

$$\begin{aligned}
l_c^*(x, u_l^k, \nu_l^k) &= f^T u_l^k + \nu_l^{kT} u_l^k [x^k - x], \\
u_l^k &= u_l(x^k), \quad \text{where } u_l(x^k) \text{ is a solution of } K(x^k)u_l(x^k) = f_l, \\
\nu_l^k &= \nu_l(x^k) = \left(u_l^{kT} K_1 \quad u_l^{kT} K_2 \quad \dots \quad u_l^{kT} K_n \right)^T,
\end{aligned} \tag{8}$$

where $x^k \in \{0, 1\}^n$ is the integer part of a solution of the relaxed master problem at iteration k . The function l_*^f is defined as

$$l_*^f(x, \lambda_l^k, \phi_l^k) = -\phi_l^{kT} \lambda_l^k x - f_l^T \lambda_l^k, \tag{9}$$

with $\lambda_l^k \in \mathbb{R}^d$ a solution of the feasibility problem $H_l(x^k)$, given by

$$\begin{aligned}
H_l(x^k) = & \underset{\lambda \in \mathbb{R}^d}{\text{minimize}} && 0^T \lambda \\
& \text{subject to} && \lambda^T K(x^k) \lambda - f_l^T \lambda < 0, \\
& && f^T \lambda - U \geq 0,
\end{aligned} \tag{10}$$

and $\phi_l^k = \left(\lambda_l^{kT} K_1 \quad \lambda_l^{kT} K_2 \quad \dots \quad \lambda_l^{kT} K_n \right)^T$. U is a valid upper bound for the objective value of the solution x^* of the original problem (3). All the results stated in ([11], [10]), used the following special notation rule, which we continue to use in this article.

Notation 1. *The expression*

$$\nu^{kT} u^k [x^k - x] = \nu^{kT} u^k x^k - \nu^{kT} u^k x,$$

should be understood in the following way. Each product of three terms

$$\nu^T u x, \text{ with } \nu \in \mathbb{R}^{nd}, u \in \mathbb{R}^d, x \in \mathbb{R}^n$$

is interpreted as

$$\nu^T u x = \sum_{j=1}^d \sum_{k=1}^n \nu(d[k-1] + j) u(j) x(k).$$

In particular, after the expression of ν^k in (8), the $\nu^{kT} u^k x^k$ term is interpreted throughout the article as

$$\nu^{kT} u^k x^k = \sum_{j=1}^n x_j^k u^{kT} K_j u^k.$$

Remark 1. The expression $\nu_l^{kT} u_l^k$ corresponds in fact to the gradient of the compliance function $\nabla_x c_l(x)$. This means that the GBD cuts $l^*(x, u_l^k, \nu_l^k) \leq y$ given by (8) are linear constraints approximating locally the compliance functions in the point (design) x^k . As a consequence, the GBD method applied to problem (2) is conceptually equivalent to the application of the Outer Approximation ([3, 7]) method to problem (3) (i.e., this equivalence does not consider minor differences in the technical details of the methods).

4 Principle of the New GBD Method

To illustrate the main idea behind the method, we will show a one-dimensional example, where the principle of the new method can be explained in a simple way. Suppose we have a convex differential function $c(x)$ in a one-dimensional space, for instance in \mathbb{R} . We consider then a minimization problem for the objective function $c(x)$ over a finite set of points $\{x^1, \dots, x^n\}$ in \mathbb{R} . The classical GBD includes cuts for every point in the discrete feasible set obtained by the algorithm in an iterative procedure, converging in a finite number of steps and obtaining a global optimal solution. The idea behind the GBD method is to approximate the non-linear objective and constraints functions with linear functions. This situation can be seen in Figure 1.

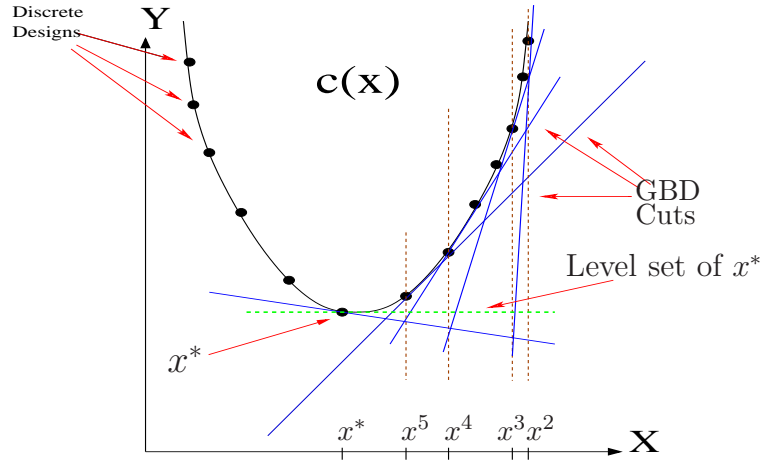


Figure 1: Classical behavior of the GBD algorithm for the minimization of a function $y = c(x)$.

The new GBD procedure here proposed possesses the same properties, but it uses the principle that it is enough to approximate the linear function at the level set of the optimal value. By using this extra information, a better performance is expected for this new algorithm. In particular, for a minimization problem in a one-dimensional space, it can be proven that the new GBD method converges in at most one iteration after having found a global optimum for the minimization problem. Note that it has not been stated that the algorithm will converge in two iterations in this type of problems, but only that it will converge at most two iterations after finding a global optimum solution. This

result does not hold for the classical GBD and Outer Approximation methods ([3]), as it has been reported in [7], where a counter example is shown. In this counter example, these methods find the global optimal solution at the first iteration, and then visit all possible points in the feasible set before converging. For the same example, the new GBD algorithm would converge exactly at the third iteration, and only visiting two feasible points before converging. This is explained graphically in Figure 2. Suppose the new GBD algorithm finds a global optimum solution x^* at iteration one. Then it will update the Master problem including the GBD cut related to x^* . The master problem is solved in iteration two, finding x^2 as solution. Instead of including the GBD cut related to x^2 , the new GBD algorithm will find a point not belonging to the discrete feasible set points, but a point in the level set of the current best design x^* (we say only current best design, because we do not know yet that it is a global optimum). In a one dimensional space, the uniqueness of this point x^U is clear for any non trivial convex objective function. The GBD cut related to x^U is included and the relaxed master problem is solved for the third iteration. The relaxed master problem solves the linear mixed integer problem showed in Figure 2. It is possible to see that the optimal solution for the relaxed master problem is (x^*, y^*) . Therefore, the discrete point x^* has appeared for the second time as a solution of the relaxed master problem. The repetition of x^* is the condition for convergence of the algorithm (see [4, 7, 11]), proving the convergence of the algorithm at the third iteration, and the optimality of x^* . Unfortunately, this property does not hold in higher dimensional spaces, but preliminary computational results shows a clear better numerical behavior for the algorithm with the new GBD method for higher dimensional problems.

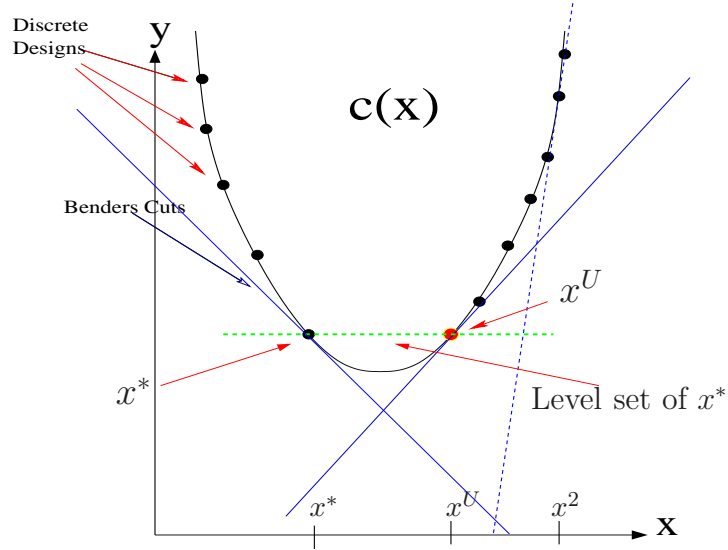


Figure 2: Optimal cuts included by the New GBD proposed algorithm for the minimization of a function $y = c(x)$. The dashed line represents the GBD related to x^2 which would have been included if performing the classical GBD or Outer Approximation method.

5 Theoretical Results

In this section, we present the theoretical basis for the new GBD method we introduce in this article, and we apply it to the minimum compliance problem (3). The ideas are simple, and aim to justify the replacement of classical GBD cuts, by cuts obtained from non feasible points. These non feasible points are not solutions of the master problem,

but belong to a subset of the design space, where the optimal integer design set is likely to be located. The projection of the continuous relaxation of the problem (3) on the design variable x is given by (see [2])

$$\begin{aligned} & \underset{x \in \mathbb{R}^n}{\text{minimize}} && \max_{1 \leq l \leq m} c_l(x) \\ & \text{subject to} && \rho^T x \leq M, \\ & && Ax \leq b, \\ & && x \in [0, 1]^n, \end{aligned} \tag{11}$$

where the functions $c_l(x)$ are defined by (4). The restriction of the problem (11) to its domain of finiteness of the compliances is convex, so any local solution of it is also a global solution (see [14]).

Definition 1. Let x^* be an optimal solution of the minimum compliance problem (3), with compliances $c_l(x^*)$, $l = 1, \dots, m$, $\max_{1 \leq l \leq m} c_l(x^*) = C^*$. Define the sets

$$X_l^0 = \{x \in [0, 1]^n : c_l(x) < C^*, \rho^T x \leq M, Ax \leq b\}$$

$$X_l^* = \{x \in [0, 1]^n : c_l(x) \leq C^*, \rho^T x \leq M, Ax \leq b\}$$

$$\tilde{X}_l^* = \{x \in [0, 1]^n : c_l(x) = C^*, \rho^T x \leq M, Ax \leq b\}.$$

We begin by stating and proving some preliminary results, which are important to justify the definition of the new type of GBD cuts.

Lemma 1.

$$\bigcap_{l=1}^m X_l^0 \cap \{0, 1\}^n = \emptyset.$$

Proof. Suppose that the set $\bigcap_l X_l^0 \cap \{0, 1\}^n$ is not empty. Then it contains at least one element x^{**} . x^{**} is feasible for the problem (3), and satisfies $\max_{1 \leq l \leq m} c_l(x^{**}) < C^*$. This is a contradiction with the optimality of x^* , proving the emptiness of the set $\bigcap_l X_l^0 \cap \{0, 1\}^n$. \square

Lemma 1 claims simply that the set $\bigcap_{l=1}^m X_l^0$ does not contain any feasible point for problem (3). All feasible 0 – 1 points belong necessarily to its complement set $\bigcap_l X_l^{0c}$. Naturally, the same applies for the optimal solutions of problem (3). A natural consequence of this is stated in the following lemma.

Lemma 2. The set of all solutions of the minimum compliance problem (3) is given by

$$\tilde{X}_c^* = \{x \in \bigcup_{l=1}^m \tilde{X}_l^* \cap \{0, 1\}^n : \max_{1 \leq l \leq m} \{c_l(x)\} = C^*\}.$$

Proof. Lemma 2 is a corollary of Lemma 1. \square

Lemma 2 claims that the solutions of problem (3) correspond to the integer points lying on the level set of the optimal compliance. In other words, Lemmas 1 and 2 imply that if we want to solve problem (3), in reality, it would be enough to search on a smaller subset of the design space. The following result is more specific about this.

Lemma 3. x^* is a solution of the problem (3), if and only if x^* is solution of the problem

$$\begin{aligned} & \underset{x \in \mathbb{R}^n}{\text{minimize}} && \max_{1 \leq l \leq m} \{c_l(x)\} \\ & \text{subject to} && x \in \bigcup_{l=1}^m \tilde{X}_l^* \cap \{0, 1\}^n. \end{aligned} \quad (12)$$

Proof. Lemma 3 is just a corollary of Lemma 2. \square

Lemma 3 states that problem (12) has the same solution set as problem (3). Problem (12) has however a much smaller feasible set. Therefore, it would be more convenient to attack this problem instead of the original version of the problem.

In [10], it was pointed out that by including the cut(s) related to a solution of the continuous relaxation (11), the convergence of the GBD algorithm can be significantly accelerated. Let us denote x^r , any solution of the continuous relaxation (11).

Lemma 4. Consider the load f_l . Let $x \in \{x \in [0, 1]^n : Ax \leq b, \rho^T x \leq M, c_l(x) > U\}$. The function $g_x^l : [0, 1] \rightarrow \mathbb{R}^+$ given by $g_x^l(\alpha) = c_l(\alpha x^r + (1 - \alpha)x)$ is a convex function. In addition, for any $c_0 \in [c_l(x^r), c_l(x)]$, we can find $\alpha_0 \in [0, 1]$ such that $g_x^l(\alpha_0) = c_0$.

Proof. The convexity of g_x^l follows from the convexity of $c_l(x)$ (see [14]). The existence of α_0 is a direct consequence of the intermediate value theorem (see any basic calculus or analysis text). \square

Definition 2. Define the sets

$$\begin{aligned} X_l^U &= \{x \in [0, 1]^n : c_l(x) \leq U, \rho^T x \leq M, Ax \leq b\}, \\ \tilde{X}_l^U &= \{x \in [0, 1]^n : c_l(x) = U, \rho^T x \leq M, Ax \leq b\}, \end{aligned}$$

where U is a valid upper bound for the optimal compliance value, i.e. $U \geq C^*$.

Lemma 4 allows us to find, for any design x^k such that $c_l(x^k) > U$, $l \in \{1, \dots, m\}$, a scalar $\alpha_l^k \in [0, 1]$ and a design $x_{l,U}^k \in \tilde{X}_l^U$ satisfying $c_l(x_{l,U}^k) = g_{x^k}^l(\alpha_l^k) = U$. Thus, if at a certain stage, a solution of the master problem (x^k, y^k) does not satisfy $\max_l \{c_l(x^k)\} \leq U$, by using Lemma 4, we can find, for every l such that $c_l(x^k) > U$, a (non feasible) design $x_{l,U}^k \in \tilde{X}_l^U$. To ensure the convergence of the problem in a finite number of steps, we need to guarantee that the integer part of the solution the master problem x^k can not be a solution of the following master problems.

In [10], it was indicated that l_c^* -cuts coming from low compliance designs are stronger and lead to faster convergence. Here we present a result in the same direction, showing that if we include the cut related to the design $x_{l,U}^k$ (of lower compliance), we do not need to include a cut related to x^k .

Proposition 1. Consider an upper bound U for the optimal compliance of the problem (3). Suppose that $(x^k, y^k) \in \{0, 1\}^n \times \mathbb{R}$ is a feasible design for the master problem (7), and such that $c_{l_0}(x^k) > U$, for some $l_0 \in \{1, \dots, m\}$. It exists $x_{l_0,U}^k \in \tilde{X}_{l_0}^U$ such that $x_{l_0,U}^k = \alpha_{l_0}^k x^r + [1 - \alpha_{l_0}^k] x^k$ and $c_{l_0}(x_{l_0,U}^k) = U$, for some $\alpha_{l_0}^k \in [0, 1]$. Then, any master problem of the type of (7), including the cut $l_c^*(x, u_{l_0}^k, \nu_{l_0}^k) \leq y$, with $u_{l_0}^k = u_{l_0}^k(x_{l_0,U}^k)$, and $\nu_{l_0}^k = \nu_{l_0}^k(x_{l_0,U}^k)$ given by (8), can not have (x^k, y) , with $y \in \mathbb{R}$, as a solution.

Proof. The existence of $x_{l_0,U}^k$ is guaranteed by Lemma 4. Suppose that a given master problem including the cut

$$l_c^*(x, u_{l_0,U}^k, \nu_{l_0}) = \nu_{l_0}^{kT} (x - x_{l_0,U}^k) + f^T u_{l_0}^k \leq y$$

possesses (x^k, y^k) as a solution, with $c_{l_0}(x^k) > U$ for some $l_0 \in \{1, \dots, m\}$. Then, we need to see that $\nu_{l_0}^{kT} (x^k - x_{l_0,U}^k) = g'(\alpha_{l_0}^k) \cdot (0 - 1)$ (see [2]). Since $g'(\alpha) < 0 \forall \alpha \in [0, 1]$, then $g'(\alpha_{l_0}^k) \cdot (0 - 1) > 0 \implies \nu_{l_0}^{kT} (x^k - x_{l_0,U}^k) > 0$. Since $f^T u_{l_0}^k = U$, we have as a consequence that

$$y^k > U,$$

which can not occur, since y^k , as a solution of the master problem, is necessarily a valid lower bound of the optimal compliance (i.e., $y^k \leq C^* \leq U$). This proves that (x^k, y^k) , $y^k \in \mathbb{R}$, can not be solution of such a master problem. \square

Proposition 1 tells us that if $\max_l \{c_l(x^k)\} > C^*$, then by including the non feasible cut

$$l_c^*(x, u_{l_0}^k, \nu_{l_0}^k) \leq y$$

in the master problem, the design x^k can not be part of a solution of any of the following master problems. Notice as well that this proposition holds independently on whether the considered design x^k is feasible for the equilibrium equations (1) or not. This implies that we do not need to include feasibility cuts of any type.

Proposition 2. *Consider the semi-infinite master problem*

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\ & \text{subject to} && l_c^*(x, u_l(z), \nu_l(z)) \leq y, \quad \forall z \in \bigcup_l \tilde{X}_l^*, \\ & && l = 1, \dots, m, \\ & && \rho^T x \leq M, \\ & && Ax \leq b, \\ & && x \in \{0, 1\}^n. \end{aligned} \tag{13}$$

Any optimal solution $(x^k, y^k) \in \{0, 1\}^n \times \mathbb{R}$ of (13), is also an optimal solution of (3).

Proof. First, suppose $\max_l \{c_l(x^k)\} > C^*$. It follows that there exists $l_0 \in \{1, \dots, m\}$ such that $c_{l_0}(x^k) > C^*$. Then, after Lemma 4, we can find $\alpha_{l_0}^k \in [0, 1]$, such that $x_{l_0}^k = \alpha_{l_0}^k x^k + [1 - \alpha_{l_0}^k] x^k$, and $c_{l_0}(x_{l_0}^k) = C^*$. This means that $x_{l_0}^k \in \tilde{X}_{l_0}^*$. As a consequence, the cut $l_c^*(x, u_{l_0}^k(x_{l_0}^k), \nu_{l_0}^k(x_{l_0}^k)) \leq y$, is already present in the problem (13). Proposition 1 with $U = C^*$ ensures that x^k can not be a solution of the master problem, obtaining a contradiction. Then, Lemma 1 states that $\max_l \{c_l(x^k)\} \not\leq C^*$. Then, the only possibility left is $\max_l \{c_l(x^k)\} = C^*$, in which case x^k is a solution of (3). \square

Problem (13) is a semi-infinite problem, and it requires us to know in advance the optimal value C^* . This makes it impossible to treat this problem as stated. As an alternative, we can consider any valid upper bound U for the compliance function. Therefore, we will look instead at the semi-infinite master problem

$$\begin{aligned}
& \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\
& \text{subject to} && l_c^*(x, u_l(z), \nu_l(z)) \leq y, \quad \forall z \in \bigcup_l \tilde{X}_l^U, \\
& && l = 1, \dots, m, \\
& && \rho^T x \leq M, \\
& && Ax \leq b, \\
& && x \in \{0, 1\}^n,
\end{aligned} \tag{14}$$

which we can treat with a relaxation process. We have the following result.

Proposition 3. *Consider a finite relaxation of the semi-infinite master problem (14)*

$$\begin{aligned}
& \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\
& \text{subject to} && l_c^*(x, u_{l_j^k}(x_{l_j^k, U}^k), \nu_{l_j^k}(x_{l_j^k, U}^k)) \leq y, \quad \begin{aligned} & x_{l_j^k, U}^k \in \bigcup_l \tilde{X}_l^U, \\ & k = 1, \dots, N, \\ & j \in \{1, \dots, n^k\}, \end{aligned} \\
& && \rho^T x \leq M, \\
& && Ax \leq b, \\
& && x \in \{0, 1\}^n,
\end{aligned} \tag{15}$$

for given numbers $N, n^k, k = 1, \dots, N$. Consider the sequence of solutions of the problem (15), given by $(x^1, y^1), \dots, (x^N, y^N)$, and the corresponding $1, \dots, N$ l_c^* -cuts respectively. If a given solution (x^{k^*}, y^{k^*}) in the sequence satisfies $\max_{1 \leq l \leq m} \{c_l(x^{k^*})\} = U$ and $U - y^N \leq \epsilon$, with $\epsilon \geq 0$, then $U \leq C^* + \epsilon$, and x^{k^*} is an ϵ -optimal solution of (3).

Proof. We start by pointing out that (15) corresponds to take the problem (14), and consider a finite number of elements in $\bigcup_l \tilde{X}_l^U$, and the l_c^* -cuts related to these elements, each of them considering only some of the load conditions.

Then, since y^N is the optimal value of (15), it satisfies the set of inequalities

$$l_c^*(x, u_{l_j^k}^k, \nu_{l_j^k}^k) \leq y^N, \quad \forall k, l_j^k.$$

These inequalities can be rewritten, after (8), as

$$f^T u_{l_j^k}^k + \nu_{l_j^k}^{kT} u_{l_j^k}^k [x_{l_j^k, U}^k - x] \leq y^N, \quad \forall k, l_j^k. \tag{16}$$

It is known (see [2]) that $\nu_l^k(x')^T u_l^k(x')x = -\nabla_x c_l(x')^T x$ for any $x' \in [0, 1]^n$ such that the equations $K(x')u = f_l$ possess a solution. It follows that (16) is equivalent to

$$c_{l_j^k}(x_{l_j^k, U}^k) + \nabla_x c_{l_j^k}(x_{l_j^k, U}^k)^T (x - x_{l_j^k, U}^k) \leq y^N, \quad \forall k, l_j^k.$$

Suppose that x^{k^*} is not a ϵ -optimum of (3). It follows that there exists $x^{**} \in \{0, 1\}^n$ feasible for (3), satisfying $\max_l \{c_l(x^{**})\} = C^{**} < U - \epsilon$ and $C^{**} < y^N$.

Since the functions $c_l(x)$ are convex functions (see [14]), they satisfy

$$C^{**} \geq c_l(x) + \nabla_x c_l(x)^T (x^{**} - x), \quad \forall x \in [0, 1]^n : c_l(x) < \infty.$$

In particular, this last condition is satisfied by the finite set of feasible designs $\{x_{l_j^k, U}^k, k = 1, \dots, N, j \in \{1, \dots, n^k\}\}$

$$C^{**} \geq c_{l_j^k}(x_{l_j^k,U}^k) + \nabla_x c_{l_j^k}(x_{l_j^k,U}^k)^T (x^{**} - x_{l_j^k,U}^k), \quad k = 1, \dots, N, j = 1, \dots, n^k,$$

and we have that the pair (x^{**}, C^{**}) satisfies all the constraints of the problem (15). It is thus, a feasible point for this problem. Since $C^{**} < y^N$, this is a contradiction with the optimality of (x^{k^*}, y^N) for the problem (15). This proves that such x^{**} does not exist, and therefore $U < C^* + \epsilon$ and so, x^{k^*} is an ϵ -optimum for (3). \square

Proposition 4. *Consider a sequence, indexed by N , of finite relaxations of the semi-infinite master problem (14)*

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\ & \text{subject to} && l_c^*(x, u_{l_j^k}^k(x_{l_j^k,U}^k), \nu_{l_j^k}^k(x_{l_j^k,U}^k)) \leq y, \quad \begin{aligned} & x_{l_j^k,U}^k \in \bigcup_l \tilde{X}_l^U, \\ & k = 1, \dots, N, \\ & j \in \{1, \dots, n^k\}, \end{aligned} \\ & && \rho^T x \leq M, \\ & && Ax \leq b, \\ & && x \in \{0, 1\}^n, \end{aligned} \tag{17}$$

$\{n^k\}_k$ is a given sequence of integers satisfying $1 \leq n^k \leq m$, and $u_{l_j^k}^{k^1} \neq u_{l_i^k}^{k^2}$ for all $(k^1, j) \neq (k^2, i)$. If $\{(x^N, y^N)\}_N$ is the sequence of solutions the master problems, and $U > C^*$, then there exists $N_0 \in \mathbb{N}_+$ such that

$$\max_l \{c_l(x^{N_0})\} < U.$$

Proof. After Proposition 1, each solution $\{(x^N, y^N)\}_N$ of the master problem such that $x^N \notin \bigcup_l X_l^U$ is prevented to be a solution of the master problem again on next iterations, just by including the cut $l_c^*(x, u_{l_0^N}^N, \nu_{l_0^N}^N) \leq y$, for some $l_0^N \in \{1, \dots, m\}$. Since the number of integer designs in $\{0, 1\}^n \setminus \bigcup_l X_l^U$ is finite, as well as the number of designs in $\{0, 1\}^n \cap \bigcup_l \tilde{X}_l^U$, there will be necessarily an $N_0 > 1$, for which none of the points in $(\{0, 1\}^n \setminus \bigcup_l X_l^U) \cup \bigcup_l \tilde{X}_l^U$ can be part of a solution of the master problem. The solution (x^{N_0}, y^{N_0}) of this master problem must therefore satisfy $x^{N_0} \in \{0, 1\}^n \cap (\bigcup_l X_l^U \setminus \bigcup_l \tilde{X}_l^U)$, in which case we have

$$\max_l \{c_l(x^{N_0})\} < U.$$

\square

Propositions 2 and 4 state that if we have a strict upper bound U^1 for the optimal objective value of problem (3) (i.e, $U^1 > C^*$), we can build an algorithm based on the relaxation of a master problem on $\tilde{X}_c^{U^1}$. When enough cuts have been added to the relaxed master problem, the solution (x^k, y^k) of the master problem will satisfy $\max_l \{c_l(x^k)\} = U^2 < U^1$.

When this happens, we can update all the cuts to the set $\tilde{X}_c^{U^2}$ through Lemma 4. The other possibility is that $U^1 = C^*$, in which case the algorithm will converge in a finite number of steps. Therefore, the only requirement is a valid upper bound for the optimal compliance, which can be obtained and updated during the algorithm. If at any stage of the algorithm, the value C^* is attained as upper bound U , then the sets $\bigcup_l \tilde{X}_l^U$ and $\bigcup_l \tilde{X}_l^*$

are the same set. At this point, when enough cuts at the level set $\bigcup_l \tilde{X}_l^*$ are added, the algorithm converges, obtaining an optimal solution.

Remark 2. *The validity of the new type of GBD cuts is supported on the fact that there is no essential difference between this new type of cuts and the ones used in a classical GBD algorithm except for the fact that the points used to build the cuts may not be feasible. This feasibility issue has no relationship with the validity of the cuts, since the validity of the cuts is only related to the correct approximation and underestimation of the original non-linear objective function by supporting planes, independently of whether these supporting planes were built from feasible points or not.*

6 Statement of the Method

In this section, we present formally the algorithm executing the new Generalized Benders' Decomposition method proposed in this article, for solving problem (2). The assumptions considered are Assumptions (A-1)-(A-4). The algorithm can be applied for reinforcement or pure topology optimization problems, as well as single, or multiple load problems.

Algorithm 1: New Generalized Benders' Decomposition approach, for the minimum compliance problem (2).

1. Set $P = Q = 1$, the upper bound $U = +\infty$, the lower bound $y^0 = -\infty$ and the convergence tolerance $\epsilon \geq 0$.
2. Compute the solution x^r of the continuous relaxation of the minimum compliance problem (3). Compute the compliances $c_1(x^r), \dots, c_m(x^r)$. Do for all load cases $l = 1 \dots, m$: {Compute the vectors $u_l(x^r), \nu_l(x^r)$ and the corresponding compliance cuts $l_c^*(x, u_l(x^r), \nu_l(x^r)) \leq y$, after (8).}
3. Solve the first relaxed master problem

$$\begin{aligned}
& \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\
& \text{subject to} && l_c^*(x, u_l(x^r), \nu_l(x^r)) \leq y, \quad l = 1, \dots, m, \\
& && Ax \leq b, \\
& && \rho^T x \leq M \\
& && x \in \{0, 1\}^n,
\end{aligned} \tag{M1}$$

by any solver for linear-mixed integer programming. If (M1) is infeasible, then stop and exit. Problem (2) is infeasible. Otherwise, denote the solution of (M1) found by (x^1, y^1) and set $P = 1$.

4. Do for all loads $l = 1, \dots, m$: {Compute the compliance $c_l(x^P)$ }. If $\max_{1 \leq l \leq m} \{c_l(x^P)\} < U$, set $U = \max_{1 \leq l \leq m} \{c_l(x^P)\}$, and set the solution index $i^* = P$.
5. Do for all $k = 1, \dots, P$: { Set $j = 1$. Do for all loads $l = 1, \dots, m$: {If $c_l(x^k) > U$, set $l_j^k = l$, set $j \leftarrow j + 1$, and compute, by a bisection procedure, $\alpha_{l_j^k}^k \in [0, 1]$ such that $g_{x^k}^{l_j^k}(\alpha_{l_j^k}^k) = U$, where g_x^l is the function defined in Lemma 4. Compute the design $x_{l_j^k, U}^k$ as $x_{l_j^k, U}^k = [1 - \alpha_{l_j^k}^k]x^k + \alpha_{l_j^k}^k x^r$. Find the displacement field $u_{l_j^k}^k$ and $\nu_{l_j^k}^k$, after (8)} }

6. Solve the relaxed master problem:

$$\begin{aligned}
& \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\
& \text{subject to} && l_c^*(x, u_{l_j}^k, \nu_{l_j}^k) \leq y, \quad k = 1, \dots, P, \\
& && j \in \{1, \dots, n^k\}, \\
& && \rho^T x \leq M, \\
& && Ax \leq b, \\
& && x \in \{0, 1\}^n,
\end{aligned} \tag{18}$$

where $l_c^*(x, u_{l_j}^k, \nu_{l_j}^k)$ given by (8), and n^k is the number of load conditions holding $c_l(x^k) > U$. Solve (18), by any solver for linear-mixed 0 – 1 programming. If (18) is infeasible, then stop and exit. Problem (2) is infeasible. Otherwise, set $P \leftarrow P + 1$, and denote the solution of this program (x^P, y^P) .

7. If $U - y^P \leq \epsilon$, then stop. The optimal design found is x^{i^*} and its optimal value U . Otherwise return to step 4.

Remark 3. *The bisection procedure included in step 5 of the algorithm supposes that the resolution of the subproblem and/or evaluation of the objective value is computationally cheap. For large-scale problems, the bisection procedure might result in a very slow algorithm, and might dominate the computational time for the complete GBD algorithm. For these cases the bisection procedure should be replaced by a more efficient line search algorithm, such as a Newton type algorithm or similar.*

7 Convergence to Global Optima

Theorem 1. *The presented Generalized Benders' Decomposition (GBD) method Algorithm 1 converges in a finite number of iterations to the global optimum value of problem (3).*

Proof. The convergence in a finite number of steps is based on the finiteness of the set $\{0, 1\}^n$. After Proposition 1, every solution (x^k, y^k) of the master problem such that $x^k \notin \bigcap_l X_l^U$ is prevented to be a solution of the master problem in the following iterations, by including the cut $l_c^*(x, u_{l_j}^k, \nu_{l_j}^k) \leq y$. Otherwise, if $x^k \in \bigcap_l X_l^U \setminus \bigcap_l \tilde{X}_l^U$, then $\max_l \{c_l(x^k)\} = \tilde{U} < U$. By setting $U \leftarrow \tilde{U}$, and consequently, updating the sets X_l^U to the sets $X_l^{\tilde{U}}$, $l = 1, \dots, m$, we have that now x^k belongs to $\bigcap_l X_l^U$, and x^k is prevented from being a solution of the following master problems, unless x^k is an optimal solution of (3) (as it was proven in [11]). Since we are discarding at each iteration, at least one design (unless optimal), the number of iterations is limited by the cardinality of the feasible set of problem (3). This number is finite, proving the convergence in a finite number of steps. Therefore, the algorithm will stop the latest, when a design $x^k \in \bigcap_l \tilde{X}_l^U$ is repeated in the sequence of solutions of the master problem. Proposition 3 ensures that, if the algorithm converges, it converges to a global optimum. \square

Corollary 1. *The presented GBD method Algorithm 1 converges in a finite number of iterations to the global optimum value of problem (2)*

Proof. The projection theorem ensures that problems (2) and (3) have the same optimal objective value. Therefore, the GBD algorithm will converge to the common global optimal value of these two problems. \square

8 Implementation

The Generalized Benders Decomposition method presented in this article is implemented in the numerical environment and high level programming language Matlab ([9]), for solving 2-D truss topology design problems. The solver used in the treatment of the master problem is the commercial branch-and-cut solver for integer programming CPLEX version 9 ([5]). The continuous relaxation (11) is solved using the Method of Moving Asymptotes (MMA), see [13, 15].

9 Numerical Examples

In this section, we present some numerical examples for the Generalized Benders' Decomposition (GBD) method proposed in this article. We will use some of the numerical examples presented in ([10]) for benchmarking the proposed method. These examples are 2-D truss design problems, where problem (3) was attacked for a particular geometry of the design space, considering different parameterizations. The design problem corresponds to finding an optimal configuration of bars forming a truss structure. The design decision considers as well the optimal area (out of a discrete set of candidate areas $\{a_1, \dots, a_{n_0}\}$) for each of the bars included in the structure. For more details about the example description, see [10]. The tolerance for the feasibility of all constraints is set exactly as in [10] (10^{-5}). All examples were run on an UltraSPARC IV processor, running at 1800 MHz, as in [10].

Name	Areas	DV	η_s	Load Cases	Load	Weight limit (M)
P1	1	74	0.0	1	1.0	23.5
P2	1	74	1e-2	1	1.0	23.5
P3	5	370	0.0	1	1.0	16.0
P4	5	370	1e-2	1	1.0	16.0
P5	5	370	0.0	3	10.0	23.5
P6	5	370	1e-2	3	10.0	23.5

Table 1: Problem statistics for the 6 problem instances solved in [10].

In Table 1, the parameterization describing the examples are given for the six examples. This parameterization scheme includes the number of candidate areas, the number of design variables (DV), the value of the re-enforcement parameter (η_s), the number of load cases in the example, the magnitude of the load, and the weight limit for the structure.

The stopping criteria for the algorithm was set exactly as in [10], i.e. as $Gap < 0.5\%$, where Gap is defined as the relative optimality gap

$$Gap := \frac{U - y^k}{U} \times 100,$$

U is the current best objective value found, and y^k is the optimal objective for the master problem at iteration k .

P	Max(C_j)	GBD iter.	Gap (%)	CPU [h:m]
P1	0.08951	1273	2.619	72:36
P2	0.08773	180	0.442	00:24
P3	0.11471	675	3.981	97:56
P4	0.11082	534	0.500	03:17
P5	1.57446	120	1.621	93:57
P6	1.53289	120	0.413	25:37

Table 2: Statistic obtained for examples P1 – P6, for the standard GBD algorithm. Three out of the six examples converged to optimality.

P	Max(C_j)	GBD iter.	Gap (%)	CPU [h:m]
P1	0.08951	1140	0.981	75:04
P2	0.08773	60	0.438	03:14
P3	0.11338	145	0.479	07:45
P4	0.11065	83	0.487	05:56
P5	1.56377	90	0.362	82:37
P6	1.53252	30	0.391	07:49

Table 3: Statistic obtained for examples P1 – P6, from [10]. The use of heuristics improves the performance of the GBD algorithm. Five out of the six examples converged to optimality.

First we ran the examples with the standard GBD algorithm based on the algorithm presented in [11]. The results of this experiment are shown in Table 2. We note that three out of the six examples converged within the time limit of 72[h]. The three examples that converged are exactly the examples with a non zero re-enforcement parameter (i.e., $\eta_s > 0$). This result shows that re-enforcement problems are better suitable to be solved by a GBD approach. This observation was already conjectured in [10], and here we confirm once more the same statement by the numerical evidence of the presented examples.

In table 3 we can see the results obtained in [10], where a standard GBD approach was used, including a heuristic preprocessing stage, for obtaining many good quality initial cuts. That algorithm considers as well two sets of optimality tolerances and time limits for the master problem, and it includes the cuts obtained from solutions of the continuous relaxation (11). These two techniques help accelerating the convergence of the algorithm. The proposed method uses these two techniques too, including a set of relative tolerances for the master problem ($tol_1 = 0.1\%$, $tol_2 = 0.05\%$), where tol_2 is considered every 30 iterations. Furthermore, a set of time limits ($T_1 = 300[s]$, $T_2 = 24[h]$) is considered, where T_2 is set only every 15 iterations. In Table 4, we see the corresponding results for the proposed method.

The results show an important difference in performance, with respect to the CPU time spent for both algorithms. The new proposed algorithm was faster in all examples, with rates of CPU time going from 1/30 (example P4) to 1/2400 (example P1) .

P	Max(C_j)	GBD iter.	Gap (%)	CPU [h:m:s]
P1	0.08951	62	0.381	0:01:50
P2	0.08773	58	0.445	0:01:56
P3	0.11338	67	0.441	0:03:36
P4	0.11068	121	0.424	0:12:20
P5	1.56521	36	0.444	1:20:43
P6	1.53282	40	0.435	1:00:54

Table 4: Statistic obtained for examples P1 – P6 with the proposed GBD algorithm.

10 Final Remarks and Future Work

We have proposed a new technique based on the Generalized Benders' Decomposition method for attacking mixed-integer optimization problems. This new method considers a solution of the continuous relaxation of the mixed integer problem. Then, for each solution of the relaxed master problem, a point in the level set of the current best solution is found by a bi-section procedure. The bisection procedure is executed between the solution of the continuous relaxation and the design coming from the solution of the master problem. For large-scale problems, the resolution of the subproblem for each step of the bi-section procedure may make the method inefficient. In these cases the bisection procedure should be replaced by a more efficient line search procedure. The numerical results confirm that the best cuts to be included in the sequence of master problems are not necessarily obtained by the solutions of these master problems. The algorithm was benchmarked with an existing implementation of Generalized Benders' Decomposition for structural topology optimization, and an improvement on CPU time in at least 2 orders of magnitude, and also an important reduction in the number of iterations was observed. These promising results suggest that this method should be investigated also other types of problems, generalizing the method to a larger class of problems. For instance, problems where the non linear functions involved are formulated as constraints, instead of as the objective function, which is the case in this work.

References

- [1] J.F. Benders. Partitioning procedures for solving mixed-variables programming problems. *Numerische Mathematik*, 4(1):238–252, 1962.
- [2] M.P. Bendsøe and O. Sigmund. *Topology Optimization: Theory, Methods and Applications*. Springer-Verlag, 2nd edition, 2003.
- [3] M.A. Duran and I.E. Grossmann. An outer-approximation algorithm for a class of mixed-integer nonlinear programs. *Mathematical Programming*, 36(3):307–339, 1986.
- [4] A.M. Geoffrion. Generalized Benders decomposition. *Journal of Optimization Theory and Applications*, 10(4):237–260, 1972.
- [5] ILOG Inc. *CPLEX 9.0 Users Manual*, 2003.
- [6] R. Lazimy. Extension of the generalized Benders decomposition. *Communications in Applied Numerical Methods*, 6:195–203, 1986.

- [7] S. Leyffer and R. Fletcher. Solving mixed-integer nonlinear programs by outer-approximation. *Mathematical Programming*, 66(3):327–349, 1994.
- [8] T.L. Magnanti and R.T. Wong. Accelerating Benders decomposition: algorithmic enhancement and model selection criteria. *Operations Research*, (29):464–484, 1981.
- [9] The MathWorks, Inc. *Matlab user manual version 7.1*, 2005.
- [10] E. Munoz. *Global Optimization for Discrete Topology Design problems By Generalized Benders’ Decomposition*. PhD thesis, Department of Mathematics, Technical University of Denmark, 2010. Chapter 4.
- [11] E. Munoz. *Global Optimization for Discrete Topology Design problems By Generalized Benders’ Decomposition*. PhD thesis, Department of Mathematics, Technical University of Denmark, 2010. Chapter 3.
- [12] W. Rei, J. Cordeau, M. Gendreau, and P. Soriano. Accelerating Benders decomposition by local branching. *INFORMS Journal on Computing*, 21(2):333–346, 2009.
- [13] K. Svanberg. Method of moving asymptotes - a new method for structural optimization. *International Journal for Numerical Methods in Engineering*, 24(2):359–373, 1987.
- [14] K. Svanberg. On the convexity and concavity of compliances. *Structural and Multidisciplinary Optimization*, 7(1-2):42–46, 1994.
- [15] K. Svanberg. A class of globally convergent optimization methods based on conservative convex separable approximations. *SIAM Journal on Optimization*, 12(2):555–573, 2002.
- [16] P. Wentges. Accelerating Benders’ decomposition for the capacitated facility location problem. *Mathematical Methods of Operations Research*, 44(2):267–290, 1996.

Article 4

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Discrete Multi-Material Optimization: Combining Discrete and Continuous Approaches for Global Optimization

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Abstract

Composite laminate lay-up design problems may be formulated as discrete material selection problems. Using this modeling, we state standard minimum compliance problems in their original Mixed-Integer Problem (MIP) formulation. We use different techniques for continuous and discrete optimization, and a Generalized Benders' Decomposition algorithm is used obtaining globally optimal solutions. The convergence of GBD is improved by using information obtained from heuristic procedures. We use an efficient heuristic technique, which is very likely to find close-to optimal solutions. This technique consists in solving a related sub-MINLP problem, based on the solution to the continuous relaxation of the original MINLP optimization problem. This sub-MINLP problem corresponds to the original mixed-integer problem, where a large number of variables are fixed (up to 90%). Solving the resulting problem is often easier and typically requires significantly less computational effort. A number of numerical examples in design of composite laminated structures is presented. Several of them are solved to global optimality, and in extension the strengths of the method are discussed. Numerical examples of up to 23.000 design variables are solved to global optimality.

Keywords: structural design optimization, integer optimization, global optimization, decomposition techniques, heuristics, laminated composite materials.

1 Introduction/Literature Review

Almost every structural or mechanical design problem can be formulated as an optimization problem with either continuous or integer decision/design variables. Despite of the fact that most practical design problems are discrete in nature, the vast majority of works on structural optimization focus on design problems with continuous variables. The reasons for this are many; continuous problems are (much) easier to solve, the size of manageable continuous problems is significantly larger compared to equivalent integer problems, off-the-shelf continuous large-scale optimization algorithms exist, and in addition many integer problems may be attacked heuristically using continuous approaches. Typically, the integer nature of the decision variables comes from the fact that it is not desirable or possible to allow for e.g. every imaginable bar or plate thickness, material property etc. for the design of a mechanical structure. Often the designer is restricted to choose from a set of predefined properties for the entity in question; be it a cross section from a table of available standard cross sections, or a material from a set of predefined suitable candidate materials. Problems of truly discrete nature are not necessarily suitable for continuous approaches and furthermore most continuous approaches give no guarantee or assessment about the quality of the solutions obtained, except that they yield some design improvement compared to the initial design. To perform true optimization, that is to obtain the best solution(s) (the

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set of global optimal solutions), more rigorous approaches are needed, and this is the topic of global optimization, which we consider in this paper.

The application addressed in this paper is that of having a design domain, which is subdivided into a finite number of regions. Each of these regions will be called throughout this paper, a design sub-domain. In every design sub-domain, the selection of a material from a set of given candidate materials is to be done. This formulation covers multi-material problems such as optimal composite laminate lay-up design with different candidate material as well as discrete fiber orientation problems. We propose to use a combination of exact global optimization algorithms, continuous relaxations, and heuristics to obtain guaranteed globally optimal solutions to these discrete design problems, which would not be possible to solve by either approach independently. In this paper heuristic procedures only have the purpose of assisting in finding globally optimal solutions, while global optimization methods are used to both find globally optimal solutions, and also to prove the optimality of these solutions. We use two heuristic procedures to improve the convergence rate of the global algorithm. The difficulty of proving global optimality depends strongly in the nature of each problem. In particular, convexity properties of the continuous relaxation of an optimization model gives a superlative help in accomplishment of this task. As a matter of fact, the optimal solution to the convex continuous relaxation gives a meaningful lower bound for a global optimum of the 0/1 problem.

Structural design of laminated composite structures entails decisions about the number of layers, selection of material in each layer (CFRP¹, GFRP², polymeric foam, balsa wood, etc.), orientation of orthotropic materials ($0^\circ, 45^\circ, \dots, 90^\circ$), individual layer thicknesses. In the current work we fix the number of layers as well as the layer thicknesses a priori and thus we only consider the problem of domain-wise selecting the optimal material among multiple candidates. Thus we continue along the lines of (discrete) topology optimization meaning that we work on a given fixed domain within which we want to select in each design sub-domain the optimal material from a number of given candidate materials. These were first presented by [34, 12] in the setting of three-phase topology optimization (void and two materials). Since then [35, 25] generalized the problem to include multiple (possibly orthotropic) materials to be selected among in the setting of optimal composite laminate design. In this paper the modeling of the continuous relaxation closely follows that of [36].

In contrast (sic!) to two-phase topology optimization, the design question is extended to include multiple distinct phases whereby the problem is enlarged. This design problem is a generalization of the void-solid (or two-phase) topology optimization problem and includes this problem as a special case where void is one of the “materials”. Thus, the multi-material minimum compliance problem also lacks existence of solutions in its continuum infinite dimensional form, as it is well-known for the two-phase topology optimization problem, see [26, 21, 22, 8]. For a finite element discretized design domain, this means that the optimal solution is mesh dependent. One way to obtain a well-posed problem is to introduce micro structures (i.e. composites) to the design space, or to exclude unwanted small scale features from the feasible set (see [6]). In this work we do not ensure existence of solutions through e.g. minimum length scale or composites but we include the possibility of selecting pre-defined (composite) materials from a set of candidate materials, throughout the also pre-defined spatial design sub-domain. Since the problem of mesh dependency exists for two-phase topology design, it also exists for the multi-material problem since the former is just a special case of the latter. In practice, however, it is our experience that mesh dependency does not pose too severe problems. Nevertheless, it still exists and should be taken care of in future research. Here we just briefly mention the issues related to defining, formulating and handling meaningful length scales when multiple phases are involved. These issues are, to our knowledge, not resolved yet and require further research.

As a general fact, the original formulation of discretized structural design problems falls into the category of nonlinear non-convex mixed-integer problems, where the state variables are continuous variables and the decision/design variables are integer variables. This corresponds to a so-called SAND (Simultaneous ANalysis and Design) formulation (see [10, 18, 19]). To handle this class of problems, several techniques are found in the literature. We briefly mention the branch-and-bound method ([23], [15]), the branch-and-cut method ([38]), Outer Approximation by [9], and the Generalized Benders’ Decomposition (GBD), [11]. In this work we apply the GBD method to treat directly the mixed-0/1 structural design problem in the sense described by [28, 29]. This technique was first introduced by Benders (Benders’ Decomposition, (BD), [4]), and aimed to solve linear mixed-integer problems. The

¹Carbon fiber reinforced polymer

²Glass fiber reinforced polymer

method was generalized to a particular class of nonlinear mixed-integer problems in [11] and further developments are due to [24]. In the last two decades, a large number of publications about variations and improvements of the method (specially in the BD method, as [33], [27]) and applications in industry have been published ([31], [17]). It seems that this tendency will continue in the coming years. With respect to structural optimization, [28, 29] applied GBD for the design of simple 2-D truss structures. That article and the present are up now to our knowledge, the only existing applications of GBD to structural optimization. [28, 29] pointed out the limitation of the method to solve large-scale topology optimization problems (i.e. many sub-domains and design variables) in terms of convergence within a reasonable amount of time and memory. Therefore, in the case of large-scale problems, the capacity of this and other methods of integer optimization is still limited. To solve these problems a relaxation is introduced letting the integer variables take on continuous values. The continuous variable approach typically uses penalization of intermediate variable values to obtain integer feasible solutions eventually, see e.g. [5, 6, 37]. The size (which is not necessarily related to the complexity) of the problems that we want to attack is roughly characterized by size of the analysis problem and the number of integer design variables. The aim of this paper is to increase the size of problems possible to solve to global optimality.

Organization of the Paper

In Section 2 we present the formulation of the discrete mass constrained minimum compliance problem. In Section 3 follows a description of a method to solve the discrete problem by use of the Generalized Benders' Decomposition (GBD) method. Section 4 describes the continuous relaxation of the mixed-integer problem and how it is used to improve the performance of the GBD method. The continuous relaxation is also used as part of a rounding heuristic described in Section 5. In Section 6 we present a method combining the previously described procedures and point out in what way this improves its practical and numerical performance. Section 7 describes briefly the implementation of the algorithm for computational experience. Following the presentation of the methods developed in this work, in Section 8 we demonstrate numerical examples solved by each of the methods independently as well as examples where both methods are used in combination to demonstrate the improvement gained through the combination of the methods. The results of the computational experiments are presented in Section 9. Finally, we round off with a discussion in Section 10, point to future use of the methods and conclude in Section 11.

2 Problem Formulation

Consider a (layered shell) structure $\Omega \in \mathbb{R}^3$. We aim to construct an optimization model to design a multi-material composite laminated structure. Ω is considered as a fixed design domain, where the distribution of material has to be assigned. A set of candidate materials with different mechanical and mass properties is provided, and our goal is to find, if a suitable objective function is given, the optimal distribution of the materials satisfying the imposed constraints. We assume linear elasticity for the mechanical model, which we discretize by finite elements, reducing the continuum problem to a finite-dimensional problem with degrees of freedom, $u \in \mathbb{R}^d$. Considering appropriate support/boundary conditions and a given load condition, $f \in \mathbb{R}^d$, the finite element equilibrium equations take the following form

$$K(x)u = f \quad (1)$$

where the stiffness matrix $K(x) \in \mathbb{R}^{d \times d}$ depends on the material constitutive properties as well as the (fixed domain) finite element strain-displacement relation as defined in (6). The constitutive properties are assumed to be given by Hooke's law (linear elasticity).

Given a number of predefined materials, n^c , with known constitutive properties E_i and mass density ρ_i , we want to minimize the compliance under static loading. In order to build the optimization model, a second discretization of the design domain Ω is made. This discretization is for the design problem, and it is independent of the finite element discretization. More precisely, the second discretization of Ω introduces a set of n^d design subdomains, and a material selection variable $x_{ij} \in \{0, 1\}$ is introduced to represent the selection of a given candidate material, $i \in \{1, \dots, n^c\}$, in every design domain, $j = 1, \dots, n^d$.

$$x_{ij} = \begin{cases} 1 & \text{if material } i \text{ is chosen in design domain } j \\ 0 & \text{if not} \end{cases} \quad (2)$$

A (design) subdomain may be a single layer in an element, a layer covering multiple elements, multiple layers within a single element etc. We remark that the discretization of the design domain may, or may not coincide with the finite element discretization. The total number of design variables n is given as the sum of the number of candidates defined within each design sub-domain, i.e. in general $n = \sum_{j=1}^{n^d} n_j^c$. However, if the number of candidate materials in all sub-domain is identical the number of variables is simply $n = n^d \cdot n^c$.

In each subdomain, it is required that only one material is chosen. This is enforced by the following linear equality constraints also called generalized upper bound constraints.

$$\sum_{i=1}^{n_j^c} x_{ij} = 1 \quad \forall j \quad (3)$$

In each subdomain, the design-dependent mass density is given by $\rho_j(x) = \sum_{i=1}^{n_j^c} x_{ij} \rho_i$ and consequently the total mass of the structure is

$$M(x) = \sum_{j=1}^{n^d} \rho_j(x) V_j = \sum_{j=1}^{n^d} \sum_{i=1}^{n_j^c} x_{ij} \rho_i V_j, \quad (4)$$

where V_j is the (fixed) volume of subdomain j . We consider the discrete minimum compliance, mass constrained problem given by

$$\underset{x \in \mathbb{R}^n, u \in \mathbb{R}^d}{\text{minimize}} \quad c(x) = f^T u(x) \quad (5a)$$

$$\text{(OP)} \quad \text{subject to} \quad K(x)u = f, \quad (5b)$$

$$(M(x) \leq \overline{M}) \quad (5c)$$

$$\sum_{i=1}^{n_j^c} x_{ij} = 1, \quad \forall j, \quad (5d)$$

$$x_{ij} \in \{0, 1\}, \quad \forall i, j \quad (5e)$$

where f are design independent nodal loads, $u(x)$ are the nodal displacements obtained as the solution to Equation (1) and $M(x)$ is the total mass of the structure, Equation (4), which is limited by \overline{M} . The only assumption we make with respect to \overline{M} , is that $0 < \overline{M}$, and the problem is not infeasible, neither has a trivial solution. The mass constraint is only relevant for multi-material problems where the candidate materials have different mass density. In the case of pure fiber angle selection problems (i.e. same physical material at different orientations), the mass constraint is redundant since all candidate materials in these problems have the same mass density.

3 Generalized Benders' Decomposition Applied to 0-1 Design Optimization Problems

In this section we introduce the resolution of the problem (5) by means of Generalized Benders' Decomposition (GBD); we give a brief description of the method, and introduce an important theoretical result with respect to the convergence of the algorithm to a global optimum. Then, we characterize the conditions and modes to improve and accelerate the practical convergence of the method.

3.1 The GBD Method

In this section, we present the GBD (see [11]) method we use in this paper to attack problem (5). GBD is a known optimization algorithm for nonlinear mixed-integer problems. It is based on separating the optimization model into two sequences of simpler optimization programs. The first sequence of problems only considers the integer variables of the problems, plus a single scalar continuous variable, making a sequence of linear mixed-integer problems. The other sequence deals only with the set of continuous variables, and it is given by a special reformulation of the equilibrium equations.

In [28], a standard topology optimization in its mixed-integer formulation was studied. We consider the stiffness matrix $K(x)$ as linear in the design variables

$$K(x) = \sum_{ij} x_{ij} B_j^T E_i B_j = \sum_{ij} x_{ij} K_{ij} \quad (6)$$

where $B_j \in \mathbb{R}^{6 \times d}$ is the finite element strain-displacement matrix for subdomain j , $E_i \in \mathbb{R}^{6 \times 6}$ is the constitutive matrix for the i 'th candidate material, and $K_{ij} = B_j^T E_i B_j$ is the resulting positive semidefinite local stiffness matrix related to the design element j for the candidate material i . Under this assumption, in [28] it was proven that the GBD method applied to the minimum compliance problem given by

$$\underset{x \in \mathbb{R}^n, u \in \mathbb{R}^d}{\text{minimize}} \quad c = f^T u \quad (7a)$$

$$\text{(OP-GBD)} \quad \text{subject to } K(x)u = f, \quad (7b)$$

$$\rho^T x \leq \overline{M}, \quad (7c)$$

$$Ax \leq b, \quad (7d)$$

$$x \in \{0, 1\}^n. \quad (7e)$$

converges in a finite number of iterations to a global optimal design. Problem (5) is a particular case of (7), where a general set of linear constraints $Ax \leq b$ is replaced by the particular case of the material selection constraints $\sum_{i=1}^{n^c} x_{ij} = 1, \forall j$. As a consequence all theoretical results that hold for problem (7) hold for (5) as well.

The GBD algorithm applied to the problem (5) supposes the inclusion of two sequences of simpler optimization problems. The first is the sequence of the so-called subproblems (SP), considering the displacement field u (a continuous variable). The second is the sequence of master problems (MP), considering the design variable x (a 0-1 variable).

The subproblem corresponds to the problem (5) with the variable x fixed to a given design $x := x^k \in \{0, 1\}^n$, so the optimization problem only takes into consideration the displacement field u . Thus, the objective value c is a function of x , i.e., $c = c(x)$

$$\underset{u \in \mathbb{R}^d}{\text{minimize}} \quad c(x^k) = f^T u \quad (8a)$$

$$\text{(SP)} \quad \text{subject to } K(x^k)u = f. \quad (8b)$$

Problem (8) simply corresponds to solve the analysis problem $u^k = K(x^k)^{-1}f$ and evaluate the compliance related to the design x^k , by $c(x^k) = f^T u^k$. Notice that we are implicitly pointing out that the analysis problem possesses a unique solution. This is due to the fact that the global stiffness matrix $K(x^k)$ is positive definite, since the optimization problem (5) is not a strict topology problem, but a multi-material selection problem, which means that all candidate materials included have non-vanishing stiffness.

The master problem is defined almost exactly as it was defined in [29], and we repeat its description and notation used, adapted to the problem (5). The master problem for iteration N corresponds to the following linear mixed-0/1 problem.

$$\begin{aligned} \text{(MP)} \quad & \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} \quad y \\ & \text{subject to} \quad l_c^*(x, u^k, \nu^k) \leq y, \quad \forall k = 1, \dots, N, \\ & \quad \rho^T x \leq \overline{M}, \\ & \quad \sum_{i=1}^{n^c} x_{ij} = 1, \quad \forall j \\ & \quad x \in \{0, 1\}^n. \end{aligned} \quad (9)$$

where l_c^* is a function defined as

$$\begin{aligned} l_c^*(x, u^k, \nu^k) &= f^T u^k + \nu^{kT} [x^k - x], \\ \nu^k &= \left(u^{kT} K_{11} u^k \quad u^{kT} K_{12} u^k \quad \dots \quad u^{kT} K_{n^c n^d} u^k \right)^T, \end{aligned} \quad (10)$$

where u^k is given by the subproblem (8), and x^k is the solution of the k -th relaxed master problem.

The following explanation rule is equivalent to the ones stated in [28]. We repeat it almost exactly, since they define the notation used through the article.

Remark 1. *The notation used here is slightly different from the one used in [28, 29], where the expression ν^k was defined differently. It is important to have this in mind before comparing the equations and algorithms presented here with those in the mentioned articles.*

3.2 GBD by Level Sets

In [30], a variation of the GBD technique, named *GBD by level set cuts*, was introduced, showing a significant improvement with respect to the classical GBD algorithm. The principle of this algorithm is essentially the same, with only one difference. At each step the master problem is solved, instead of solving directly the subproblem (SP) related to the design obtained from the master problem, a more convenient design is searched. A bisection procedure allows us to find a non-integer point at the level set of the incumbent solution's objective value. Then, a GBD cut related to this non-integer point is formed and added to the master problem. This procedure is done at each iteration of the algorithm. The bisection procedure requires to have previously computed the solution of the continuous relaxation x_R^* of the problem. The sought non-integer point lies on the straight line connecting x_R^* , and the current solution of the master problem x^k . The exact design lying in the intersection of the level set of the incumbent solution and the line between x_R^* and x^k is found by the bisection procedure. In [30], all details about this technique are explained in detail. Throughout this article, we use this the improved GBD variant for all experiments. Since we do not use the classical GBD algorithm at any moment, we use the name *GBD algorithm* to refer to this improved variant of the GBD technique.

3.3 Convergence of the GBD Algorithm

The convergence of the method is measured by the Optimality gap (O. gap) of convergence, which describes the maximum relative difference between the objective value of the incumbent solution of the method, and the absolute global optimum value the problem. It is defined as

$$O.gap = \frac{1}{100} \cdot \frac{|UB - LB|}{UB} [\%]$$

It is a measure of how much the objective value can improve, and it is often used in global optimization methods. The convergence of the GBD algorithm to a global optimum was proven in [28], and it is based on the convexity of the continuous relaxation (to $[0, 1]^n$) of the projection of the compliance function $c(x, u) = f^T u(x)$ on the design variable space $c(x) = f^T K(x)^{-1} f$. The proof is rather technical and not given here.

3.4 Description of the GBD Algorithm

The GBD algorithm is briefly described in this subsection. A complete and detailed statement of the algorithm is presented in [28].

The main idea of the GBD algorithm is to approximate the projection of the nonlinear mixed-integer problem on the integer design variable. This approximation produces a linear mixed-integer problem, where all the continuous variables of the original problem have been removed by the projection operation, and only one scalar continuous variable is considered. At each iteration the algorithm adds a linear constraint which is a first-order approximation of the projected compliance function at a given design x^k . In Figure 1 it is seen how the constraints approximate the compliance function at each point visited.

It is important to remark that even if there is a theoretical convergence of the method in a finite number of iterations, there is no guarantee that this convergence is reached within reasonable CPU resources (time and memory). This is due to the fact that a master problem, which is a linear mixed-0/1 problem, takes longer and longer to be solved and consumes more memory, as more linear constraints are added. As a consequence, the size of the problem must be selected in a way such that the convergence of the method is observed in numerical experiments.

Algorithm 1 Generalized Benders' Decomposition

```
UB ← ∞
LB ← −∞
k ← 0
x0 ∈ [0, 1]n such that it is mass feasible
if x0 ∈ {0, 1}n then
    UB ← c(x0)
end if
ε > 0
while |LB − UB| > ε do
    uk, νk ← solve (SP) using xk
    lc*(x, uk, νk), see (10)
    (xk+1, zk+1) ← solve (MP) including lc*(x, uk, νk) ≤ y, see (9).
    LB = zk+1
    if c(xk+1) ≤ UB then
        UB ← c(xk+1)
    end if
    k ← k + 1
end while
```

4 Convex Continuous Relaxation

The original non-convex mixed-integer program in (5), can be reformulated in so-called nested form as an integer program with a convex objective function if the displacements u are eliminated by use of the equilibrium condition (given that the stiffness matrix $K(x)$ is non-singular, i.e. $u(x) = K(x)^{-1}f$). Thereby we obtain an equivalent integer program with a convex objective function in the design variables x only. Furthermore if the integer requirement on the design variables is relaxed, a convex continuous optimization problem is obtained

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad c(x) = f^T K(x)^{-1} f \quad (11a)$$

$$(R) \quad \text{subject to} \quad \sum_{i,j} x_{ij} \rho_i V_j \leq \overline{M} \quad (11b)$$

$$\sum_{i=1}^{n_j^c} x_{ij} = 1, \quad \forall j \quad (11c)$$

$$0 \leq x_{ij} \quad \forall (i, j) \quad (11d)$$

Note that the material selection constraints (3) ensure that the variables fulfill $x_{ij} \leq 1$. Thus there is no need for an upper bound on the variables and it is sufficient to ensure non-negative design variables. The optimization problem given by (R) is convex as shown by [39, 36], and thus a locally optimal solution x_R^* is also a global optimum of (R), and such solution may be obtained using any suitable nonlinear optimization algorithm such as the sequential quadratic programming method SNOPT [14] or the interior point method Ipopt [40]. Furthermore, (R) is a (nested form) continuous relaxation of (P) and it has a larger feasible set than (P). In other words the feasible set of (P) is a subset of the feasible set of (R). Thus,

- If the optimal solution to (R) happens to be an integer solution then it is also an optimal solution to (P).
- The optimal solution to (R) is better than or as good as the solution to (P), i.e. $c(x_R^*) \leq c(x_P^*)$. Thus, it may be used as a lower bound estimator for the original 0/1-problem.
- If there is no feasible solution to (R) then there is no feasible solution to (P) either.

The motivation for solving a convex continuous relaxation in the process of attacking the integer optimization problem is that it can be solved to global optimality with reasonable resources, and thereby

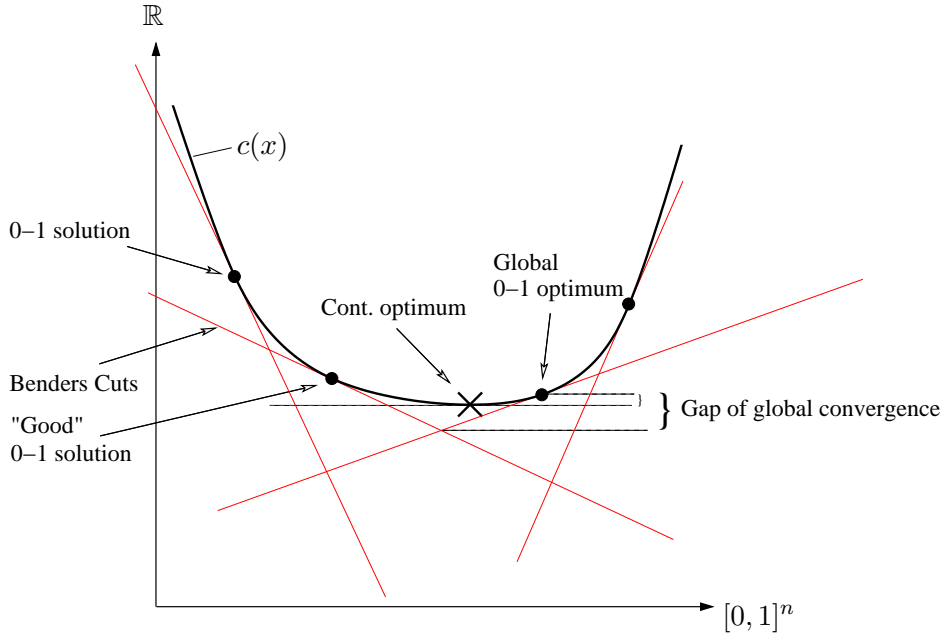


Figure 1: Conceptual illustration of the GBD algorithm on an unconstrained problem. The projection of the compliance $c(x)$ is approximated by linear functions (in red). In case of a constrained problem the continuous optimum will in general not generate a horizontal cut.

it may be used as a relatively fast way of obtaining a good lower bound assessment of the attainable performance of the original integer optimization problem. This lower bound can be used as a valid lower bound within GBD if it is better than the best valid lower bound obtained from the master problems (MP). Recall that the goal within GBD is to improve iteratively the lower and upper bound so as to close the gap between them. If a good valid lower bound can be obtained early in the solution process of GBD, the sequence of sub- and master problems the method is more likely to converge within reasonable CPU resources. Depending on the specific problem, the continuous optimum may give valuable information about the integer solution. As stated above, if the continuous solution is integer-valued it is in fact the integer optimal solution. This situation, however, is very unlikely and virtually never seen in the considered types of problems. Nevertheless, it is not unusual to see that a fraction, typically 50 – 90%, of the continuous variables attain integer values in the continuous optimum, and furthermore most domains at least have some of their variables at the lower bound, i.e. in many domains the continuous optimum has "discarded" some of the candidate materials. Note that this situation, does not mean that those materials are not part of the optimal integer solution, but it still gives information about a provenly good design, though continuous. The (reasonable) hope though is, that the solution to the integer problem is close in some sense to the continuous optimal solution. If many of the continuous variables take integer values in the optimal continuous solution, this hope is most certainly reasonable. The larger the number of continuous valued variables in the optimal continuous solution, the less reasonable this assertion is. These observations motivate the use of a heuristic procedure that can obtain a good upper bound (incumbent solution) in terms of an integer and feasible solution with a good objective function.

5 Heuristics

The use of heuristics within branch-and-bound/cut algorithms is well-known and standard nowadays. In this paper we propose to also use heuristics to enhance the rate of convergence of the GBD algorithm itself. Thus we use heuristics to obtain a good (but not necessarily optimal) integer solution early in the solution process as a sort of "warm start" of the GBD algorithm. The motivation for doing so was pointed out by [28, 29] where it was shown that the GBD algorithm may take advantage of a good initial solution (upper bound) as well as a lower bound as obtained from a relaxation.

As described in the section on GBD a (possibly long) sequence of MILPs are solved in the Master

Problems (MP). MILPs are typically solved by implicit enumeration strategies such as branch-and-bound/cut algorithms. These algorithms rely on the solution of relaxations at each node visited in the enumeration tree and on basis of this solution, branching in the tree is done. The efficiency of these algorithms relies heavily on the use of heuristics to obtain feasible solutions as well as heuristics for improving feasible solutions. State-of-the-art MILP solvers such as the commercial codes CPLEX ([20]), as well as the academic codes Gurobi ([16]), SCIP [2] employ a number of heuristics to speed up the convergence rate.

5.1 GBD-RENS

The GBD-RENS heuristic we present here is inspired by the so-called Relaxation Enforced Neighborhood Search (RENS) proposed for MILPs by [7]. The idea of this heuristic is to solve a continuous relaxation to optimality and observe which variables that attain integer values in this solution. Integer-valued variables are then fixed at their obtained value and a Large Neighborhood Search in the remaining intermediate-valued variables is performed through a sub-MINLP where only the intermediate-valued variables from the relaxation are considered (now as integer variables). Thus, we formulate the rounding heuristic as a sub-MINLP. The resulting sub-MINLP is solved using GBD on the reduced problem. The size and thereby the cost of solving the sub-problem naturally depends on the integrality of the continuous solution. Note that by solving the sub-problem to optimality we obtain *the* best rounding possible for a given continuous relaxation solution. Also, if the feasible set of the sub-MINLP turns out to be empty, no feasible rounding of the continuous solution exists, see [7]. The solution obtained for the sub-MINLP is passed back to the global problem and used as a good initial solution in the complete GBD. Also note that the continuous relaxation of the complete MINLP minimum compliance and the corresponding relaxation of the sub-MINLP problem have exactly the same solutions (KKT points of the relaxation of the MINLP are also KKT points of the relaxation of the sub-MINLP). Therefore, the continuous relaxation of the sub-MINLP problem does not give any additional information. For some examples the resulting sub-MINLP problem is not easy to solve either. This could happen if the fraction of fixed variables is not big enough, and thus there is no real advantage in attacking the sub-MINLP problem. An alternative to overcome this complication, is to introduce a variation of the GBD-RENS heuristics, which is to do a selected rounding of the solution of the continuous relaxation before fixing variables with integer values. This means that we set a threshold $\lambda_s \in [0, 0.5]$, such that each design variable with a value outside the interval $[0.5 - \lambda_s, 0.5 + \lambda_s]$ in the solution of the continuous relaxation is rounded. In exchange, it is also expected that if a variable attains a non-integer value, it is less likely that this variable will have an integer value in the solution of the original integer problem. So it is clear that the larger the number of fixed variables, the less likely it is to be able to find an optimal solution when treating the sub-MINLP problem. In addition, it is even possible that, if too many variables are fixed, the resulting sub-MINLP may not be a feasible problem. As a consequence, to make the heuristic more robust with respect to the type of problem, a second variation of the heuristic is introduced, by running the heuristic procedure iteratively, in such a way that the number of fixed variables decreases each time the GBD-RENS heuristic is executed. This iterative procedure is controlled by the parameter α_s , which is updated each time before starting it. Another aspect to take into account is the fact that this heuristic may be used to control the size of the resulting sub-MINLP we are willing to solve. This would be helpful if we want to attack a design problem of maybe 50.000 variables. Suppose that in this hypothetical example, the continuous relaxation solution obtained has for instance 35.000 0/1 values. In this case, the remaining sub-MINLP problems has 15.000 variables, which is still too big. In this case, we are able to control the size of the sub-MINLP problem by setting a value of the threshold parameter $\lambda_s < 0.5$. We could also set the minimum percentage of variables to be fixed in the sub-MINLP problem. In this way, the use of the modified GBD-RENS heuristic may lead to fix maybe 48.000 variables. Then we have a sub-MINLP of 2.000 variables, which is more likely to be successfully attacked by the GBD-RENS heuristic.

5.2 GBD-SIMP Heuristic

The strengths of the GBD algorithm for finding low objective value designs can be improved by use of a different law for interpolating the stiffness matrix, as it was done in [28, 29]. This means that the

stiffness matrix given by (6), is replaced by

$$K(x) = \sum_{ij} [\alpha x_{ij} + (1 - \alpha) x_{ij}^2] B_j^T E_i B_j \quad (12)$$

where $\alpha \in [0, 1]$ is a parameter controlling the mixture of two interpolations schemes (SIMP interpolation schemes for penalization values $p = 1$ and $p = 2$, see [5]). It is important to note that for any value of $\alpha < 1$, the continuous relaxation is not convex, and therefore the GBD method can not longer guarantee convergence to global optimality. For that reason, for these values of α , the GBD method is no more than a heuristic to find good solutions. In addition, numerical experiences show that for a low value of the mixture parameter α , the GBD algorithm converges quickly, but less chances of finding good solutions exists. Therefore, again, the use of an iterative procedure, calling this heuristic several times, updating each time the value of α , from $\alpha_0 \in (0, 1)$ to $\alpha = 1$ seems to be a robust procedure to find designs with low objective value in short time.

Remark 1. *Note that all heuristics described here produce one or several candidate designs. These solutions not only helps in the improvement of the upper bound of the optimal value, but also produce one compliance GBD cut per each of these solutions.*

6 Description of the Method

In this section we describe how we combine the previously described procedure(s) in order to solve up to medium scale sized problems on the form of (5). We describe the implementation of these combined methods, and indicate possible variations of them. A comparison of the performance of the different combinations is given in Section 8 on numerical examples. Pseudo-code for the procedure combining all the methods described is shown in Algorithm 2. As it was indicated in [28] the GBD method applied to

Algorithm 2 Overall procedure to solve (5)

```

UB ← ∞
LB ← -∞
k ← 0
(xR*, c(xR*)) ← solve continuous relaxation (R) by SNOPT
LB = c(xR*)
(xHeur1*, c(xHeur1*)) ← Apply Heuristic Strategy 1, input: xR*
⋮
(xHeurq*, c(xHeurq*)) ← Apply Heuristic Strategy q, input: xR*
UB = c(xHeurq*)
x0 = xHeurq*
Apply Algorithm 1 (including initial GBD cuts related to all found designs)

```

the minimum compliance problem (5) converges to an optimal solution in a finite number of iterations. However, in practice, this number is unknown and could potentially be very large. Furthermore, the size of the master problem grows with the number of iterations (one or more cuts added at each iteration), leading to a longer solution time for each master problem, which may prevent the algorithm to converge in a reasonable amount of time. The idea of the algorithm is to use as much (heuristic) information as possible in order to speed up the convergence rate. At any stage of the algorithm, it is possible to assess the closeness of the current solution to the global optimum. This information might be useful, depending on the order of magnitude of the gap between the best bounds obtained. If this is not the case, one could use any method that gives a better estimate of a lower bound for the global optimum.

As shown by [29] the quality of any cut is defined according to their Pareto dominance value, which depends on the objective function value of the solution generating the cut. Thus, better objective feasible solutions may in the case of the problem given by 5 be used to generate good cuts in the sense that they have a positive influence on the convergence rate compared to dominating (less good) cuts.

To sum up, the convergence rate of Benders decomposition may be improved by

1. using the solution to a convex continuous relaxation to improve the estimate of a lower bound for the global optimum. This continuous solution also generates the best possible GBD cut ([29]) that can be included in the GBD algorithm.
2. including cuts generated from good 0/1-solutions obtained by any heuristic method.

Ad 1) The solution obtained by solving the continuous relaxation (11) generates the only non-dominated cut, that is, the best cut in the sense of Pareto dominance.

Ad 2) Any heuristic method that generates good solutions may be used to improve the convergence rate in the sense that each good solution generates a cut forming a good approximation to the compliance function. Furthermore a solution obtained early by the heuristic procedures may even be a global optimal solution meaning that GBD only needs to improve the lower bound which typically also speeds up once the optimal solution is found. In this article we will test and compare four algorithms. These algorithms are based on the presented GBD algorithm and the use of heuristics. The first algorithm corresponds to the GBD method alone, executed without the combination with heuristic procedures (GBD-1, Algorithm 1). The second algorithm corresponds to the use of the GBD method in combination with the GBD-SIMP algorithm (GBD-2, Algorithm 2 with GBD-SIMP heuristic strategy). The third algorithm tested is the combination of the GBD-RENS heuristic with the GBD algorithm (GBD-3, Algorithm 3 with GBD-RENS heuristic strategy). The fourth algorithm implemented is the combination of the GBD-RENS, the GBD-SIMP and the GBD algorithm (GBD-4, Algorithm 2, with these two heuristic strategies).

7 Implementation

In this section, we describe briefly, the implementation of the algorithms described in the article in numerical experiments.

The GBD algorithm was written for the design of multimaterial composite laminated structures. The code is implemented on the MUST platform ([1]) which is an in-house research code for analysis and design of laminated composite structures. The examples are discretized using 9-node degenerated shell finite elements with 5 degrees of freedom per node (3 translational and 2 rotational), see e.g. [3, 32].

The resolution of the Master Problem was attacked using the mixed-integer optimization solver GUROBI ([16]). The continuous relaxation to the minimum compliance problem was attacked with the NLP solver SNOPT version 7.2-8 ([13]). All numerical examples were run on the Fyrkat cluster (Aalborg University, Denmark). Unless explicitly specified, all optimization parameters for the solvers are the default values.

8 Numerical Examples

In this section, we present a set of numerical examples to be solved with the proposed algorithms applied to optimal design of multimaterial (laminated) composite structures. This type of structure is often modeled as shells, and therefore, a shell finite element (FE) discretization is used to perform the static equilibrium analysis. The design discretization does not necessarily match the FE discretization, as it will be the case in many of the examples. For some examples we make use of so-called patches, which are groups of elements having the same design variable associated with them. This serves as a way of reducing the number of design variables as well as a means of providing for more manufacturing near designs in the sense that the laminates are typically produced using mats covering larger areas (i.e. multiple elements) of the structure.

Table 1 shows the general description of the set of examples included in the article. It includes the number (*Prob*) and Name (*Description*) of the problem, the number of candidate materials considered in the problem (*# Mat.*), the design discretization of the problem (*Design Discr.*) in the format $PaxbxcL$, where a, b represents the in-plane design discretization, and c represents the number of layer of the structure considered in the design problem. The field *Variables* states the total number of design variables introduced in the optimization problem. *FE Discr.* specifies the FE discretization of the problem, in the format $Edxe$, where d, e represents the finite element analysis discretization in each direction in the plane. *# LC's* stands for the number of load cases considered in the problem. Finally, \overline{M} represents the mass limit for the mass constraint of the design problem.

8.1 Examples 1-3

These first three examples illustrate the application of the proposed method to a doubly curved parabolic shell structure. All three instances are solved using an FE discretization of 32 by 32 shell elements in the plane of the structure. In all three examples the design discretization through the thickness comprises eight layers of equal and fixed thickness ($8 \cdot 0.01m$). In the plane, Example 1 has 2 by 2 design domains in each layer, Example 2 has 4 by 4 design domains per layer and Example 3 8 by 8. In all three examples the structure is subjected to one load case: a central point load acting in the vertical direction. The design task is to select the optimal material out of five possible in each domain. Four of the materials are instances of a relatively stiff orthotropic material oriented at four pre-defined directions ($-45^\circ, 0^\circ, 45^\circ, 90^\circ$) defined relative to the global x-axis. The fifth candidate material is a polymeric sandwich foam of low weight and stiffness.

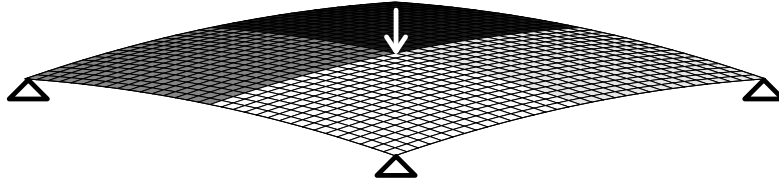


Figure 2: Example 1: sketch of parabolic shell. Geometry: base lengths $1.0 \cdot 1.0 m^2$, height $0.1 m$, shell thickness $0.08 m (= 8 \cdot 0.01 m)$, design discretization in greyscale (2×2 patches), analysis discretization (32×32 elements), vertical point load in the center and hinged support at each corner. Short notation: P2x2x8L, E32x32.

8.2 Examples 4-7

These four examples illustrate optimal discrete fiber angle orientation on a plane disc problem. All four instances are solved using an FE discretization of 32 by 32 shell elements in the plane of the structure. The disc is clamped along the left edge and subjected to a vertical downward acting point load in the lower right corner. The design discretization for the three examples is of increasing resolution in the plane of the disc, Example 4 has 4 by 4 design domains in the plane, Example 5 has 8 by 8, Example 6 has 16 by 16 and Example 7 32 by 32. The design problem is a pure fiber orientation problem, i.e. all candidate represent the same orthotropic material oriented at four ($-45^\circ, 0^\circ, 45^\circ, 90^\circ$) or twelve ($-75^\circ, -60^\circ, \dots, 0^\circ, 15^\circ, \dots, 90^\circ$) distinct directions. Thus, the example has no mass constraint (of relevance). The material properties are identical to those of the orthotropic material in Example 8-9.

8.3 Examples 8-9

In these examples we solve plane problems with two independent load cases of equal importance ($w_1 = w_2 = 0.5$) and loads with equal magnitude ($|P1| = |P2|$) acting at midspan oppositely on each face. In both load cases the plate is hinged at all corners ($u_i = 0$), see Fig. 4. The physical domain within which the material is distributed is a rectangular disc of dimension $4.0m \times 2.0m \times 0.5 \cdot 10^{-3}m$. The domain is discretized by two different meshes, (20×10) and (40×20) respectively and in each design sub domain (=element) five candidate materials are possible. The first candidate material is a light and soft material representing e.g. isotropic polymeric foam and the remaining four candidate materials represent a heavier and stiffer orthotropic material oriented at four distinct directions ($-45^\circ, 0^\circ, 45^\circ$ or 90°). We set the mass constraint such that the heavy orthotropic materials can be chosen in at most 35% of the domain. The constitutive properties in the principal material coordinate system the orthotropic material and of the foam material are given in Fig. 4.

8.4 Example 10

The following very simple example illustrates the possibility of distributing a limited amount of material through the thickness of the domain as well as in the plane. A design domain is given in terms of a

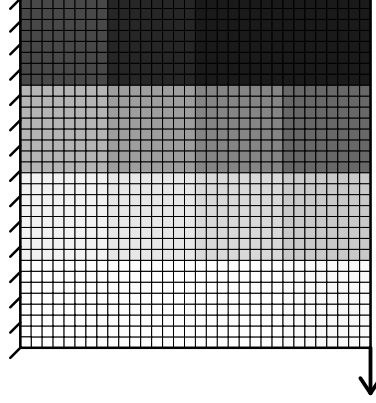


Figure 3: Example 4: sketch of clamped membrane disc. Geometry: side lengths $1.0 \cdot 1.0 \text{ m}^2$, thickness $0.5 \cdot 10^{-3} \text{ m}$, design discretization in greyscale (4x4 patches), analysis discretization (32x32 elements), vertical downward acting point load at lower right corner. Clamped (all DOFs fixed) along left edge. Short notation: P4x4x1L, E32x32.

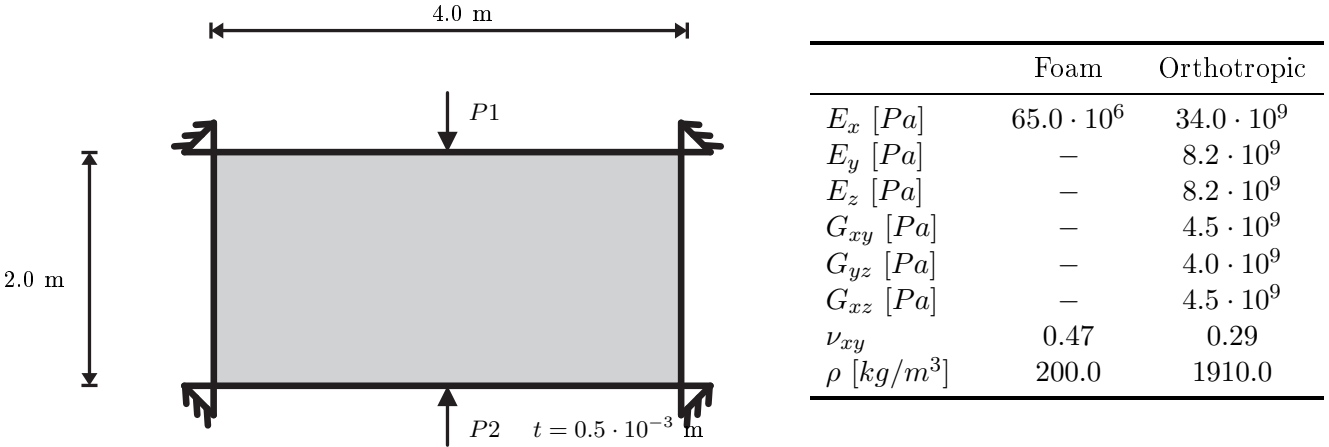


Figure 4: Example 8-9. Left: Domain geometry and boundary conditions. Loads act independently. Right: Material properties in principal material coordinate system for the candidate materials.

simply supported beam (discretized using shell elements) subjected to a uniform transverse pressure load in the vertical direction, see Fig. 5. The domain is discretized into 20 by 2 elements in the plane of the structure and five layers through the thickness. This discretization is used for the analysis as well as the design. The total volume of the design domain is 1.25 m^3 . The mass density of the lightweight candidate material is $\rho = 200 \text{ kg/m}^3$ and that of the heavy candidate material is $\rho = 1910 \text{ kg/m}^3$. Thus with a total mass constraint of 1500 kg , heavy material can not be chosen in more than 58.5% of the total design domain corresponding to 116 element layers. The material properties of the candidate materials used in this example are identical to those shown in Fig. 4.

8.5 Examples 11-13

This set of examples demonstrates the ability to perform optimal multi-layered composite plate design. We solve the same design problem using different design discretizations through the thickness to investigate the influence on the optimal design. The physical domain within which the material is distributed is a quadratic plate of dimension $1.0 \text{ m} \times 1.0 \text{ m} \times 1.0 \cdot 10^{-2} \text{ m}$. The plate is loaded at the center by a point load P and each corner is hinged ($u_i = 0$). A sketch of the problem is shown in Fig. 6. All three examples employ a (24×24) in-plane discretization. Example 11 is discretized through the thickness with 8 layers

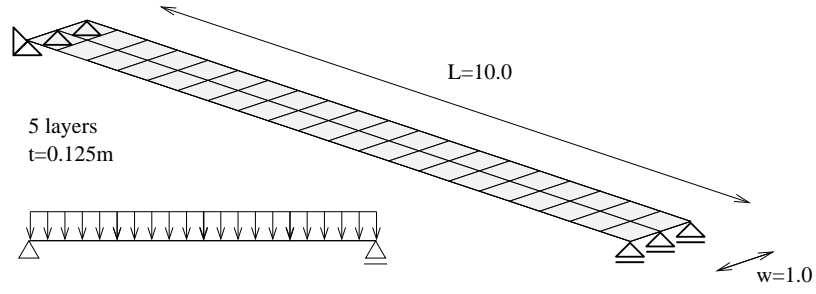


Figure 5: Example 10: Geometry: side lengths $10.0 \cdot 1.0 \text{ m}^2$, shell thickness 0.125 m ($= 5 \cdot 0.025 \text{ m}$), design discretization is identical to the analysis discretization (20×2 elements), transverse distributed pressure load and simply supported at each end.

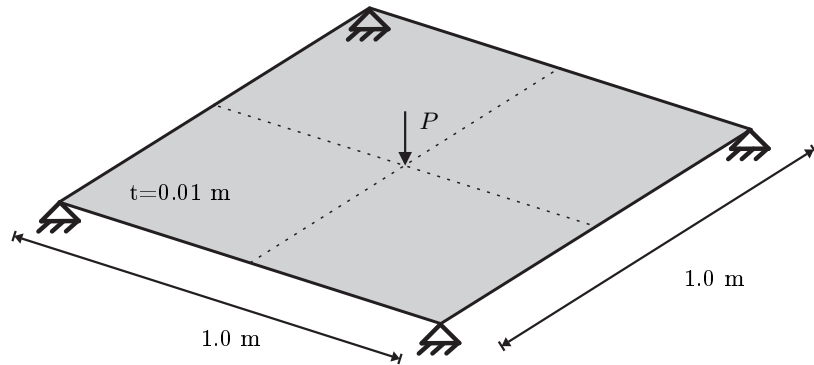


Figure 6: Example 11-13. Multi-layered (4 or 8) corner-hinged plate with point load applied at the center. See Fig. 4 for material properties.

whereas example 12 and 13 have 4 layers. The candidate materials are identical to those in the previous example, i.e. a light and soft isotropic foam material and a heavy and stiff orthotropic material oriented at four distinct directions, see Fig. 4. For more information on the problem characteristics please consult Table 1.

8.6 Computational Experience

The 13 examples were used for setting 17 computational examples (in the case of examples 4, 5, 6 and 7, two different sets of material angle candidates were considered, generating one extra numerical sub-example for each of these ones). For each of these computational examples, 4 sets of numerical experiments were carried out. The first set of examples corresponds to the execution of the GBD algorithm without considering any heuristic procedure (GBD-1). The second set of examples corresponds to the execution of the GBD algorithm where the SIMP-GBD heuristic was used in combination with the GBD algorithm (GBD-2). The third set of examples corresponds to the use of the GBD-RENS heuristic procedure in combination with the GBD algorithm (GBD-3), and the fourth set of examples is the one combining these two heuristics in combination with the GBD algorithm (GBD-4). The total CPU-time allowed for each example was 96 [h], and the algorithm is set to stop whenever the optimality gap reaches the tolerance of 1.0%. However, we consider as a satisfactory result, if the considered algorithm is able to find globally optimal solutions within an optimality tolerance of 3%. Besides, we set a maximum cpu-time of 3000[s] for the execution of each relaxed master problem. The reason for setting this limit value, is to avoid the MILP solver trying to solve to optimality eventual instances of the master problem, which are too difficult and that could take too many hours or even days to be executed.

Prob.	Description	# Mat.	Design discr.	Variables	FE discr.	# LC's	\overline{M} (kg)
1	Parabolic Shell	5	P2x2x8L	160	E32x32	1	7.248
2	Parabolic Shell	5	P4x4x8L	640	E32x32	1	7.248
3	Parabolic Shell	5	P8x8x8L	2560	E32x32	1	7.248
4	Clamped Membrane	4/12	P4x4x1L	64	E32x32	1	n/a
5	Clamped Membrane	4/12	P8x8x1L	256	E32x32	1	n/a
6	Clamped Membrane	4/12	P16x16x1L	1024	E32x32	1	n/a
7	Clamped Membrane	4/12	P32x32x1L	4096	E32x32	1	n/a
8	Two Load Case	5	E20x10x1L	1000	E20x10	2	3.0
9	Two Load Case	5	E40x20x1L	4000	E40x20	2	3.0
10	Simply sup. beam	5	E20x2x5L	1000	E20x2	1	1500.0
11	LayeredPlate	5	E24x24x8L	23040	E24x24	1	6.8
12	LayeredPlate	5	E24x24x4L	11520	E24x24	1	6.8
13	LayeredPlate	5	E24x24x4L	11520	E24x24	1	10.6

Table 1: Summary of problem characteristics for the numerical examples.

9 Results

In this section, we present the computational results for the 13 (17) examples introduced in Sect. 8. In total 68 numerical examples were executed, which correspond to the execution of the 17 numerical examples described in Sect. 8, for each of the four methods described in Sect. 8.6 (GBD-1, GBD-2, GBD-3 and GBD-4).

The results for these sets of examples is shown in Tables 2, 3, 4, 5 respectively. These tables show the information about the best objective value attained by the algorithm *Best UB*, the objective value of the continuous relaxation solution (R) (*R Sol.*), the best value of the lower bound of the optimal solution obtained by the GBD method *GBD LB*, the final optimality gap at stop *O. Gap*, and the total number of valid GBD cuts included in total in the algorithm *# GBD cuts*.

For the set of examples executed with the algorithm GBD-1, 3 examples reached a final optimality gap smaller than 1.0%, 5 examples reached a gap < 3% (including the 3 that reached 1.0%), and 10 examples reached under 5.0%. For examples run with GBD-2, 4 examples reached the stop criteria 1.0%, 8 examples reached a gap < 3.0%, and 11 were under 5.0%. For the examples run with GBD-3, 3 examples reached the stop criteria 1.0%, 6 examples reached a gap < 3.0%, and 11 were under 5.0%. Finally, for the examples run with GBD-4, 4 examples reached the stop criteria 1.0%, 11 examples reached a gap < 3.0%, and 12 were under 5.0%. In addition, to make the comparison among the different algorithms more clear, Table 6 shows the final convergence gap *O. Gap* for each group of examples.

The comparison of the results in terms of convergence (*O. gap* at stop) for each set of numerical examples is presented in Table 6.

10 Discussion

In general, the performance shown of the four algorithms is satisfactory, and shows the general strengths of the GBD algorithm itself. The use of the presented heuristics shows how the method is able to find better designs, and therefore, is able to find more tight bounds for the assessment of global optimality of the algorithm, which is important specially when treating medium-large scale problems. The combination of the two presented heuristics showed the best results in the sense of obtaining solutions with the smallest objective value, and obtaining the lowest optimality gap among the examples not reaching the stop criterion of 1.0%.

Note that the combining heuristic procedures algorithm (GBD-4) reached a negative optimality gap at convergence for examples 4.1 and 5.1. This is nothing to worry about, since these values are subjected to the optimality tolerance for the solution of the master problem, obtained by the MILP solver. These values fall inside the usual optimality tolerance of any MILP solver. Thus, these numbers are perfectly reasonable.

Prob.	Best UB	(R) Sol. LB	GBD LB	O. Gap	# GBD cuts
1	31.827	27.707	27.706	14.867	379
2	27.102	22.175	22.217	21.986	332
3	19.330	17.695	17.672	9.237	439
4.1	143.828	138.053	142.549	0.897	126
4.2	128.382	126.157	127.154	0.966	23
5.1	186.441	184.448	185.956	0.261	4
5.2	123.099	120.885	121.217	1.553	759
6.1	134.255	128.212	128.135	4.727	2029
6.2	121.809	116.903	116.879	4.197	1235
7.1	131.828	125.028	124.935	5.439	1225
7.2	117.337	113.506	113.460	3.376	479
8	568.568	22.017	22.092	2473.6	2501
9	825.281	22.718	22.672	340.19	595
10	163.273	160.178	161.354	1.189	467
11	30.108	28.760	28.691	4.688	241
12	68.842	34.841	34.725	97.587	694
13	21.356	20.529	20.520	4.024	503

Table 2: Numerical Results for GBD with out any Heuristics (GBD-1). For examples 8,9, and 12, the method failed. In general, average convergence to global optimality of 5.56% (excluding examples 8,9 and 12).

Prob.	Best UB	(R) Sol. LB	GBD LB	O. Gap	# GBD cuts
1	31.337	27.707	27.576	13.098	512
2	23.746	22.175	22.220	7.058	916
3	18.556	17.695	17.678	4.864	782
4.1	143.828	138.053	142.406	0.999	95
4.2	128.021	126.157	127.738	0.222	44
5.1	186.441	184.448	186.162	0.150	11
5.2	123.036	120.885	121.217	1.501	752
6.1	131.587	128.212	128.451	2.441	1186
6.2	120.012	116.903	116.867	2.659	1284
7.1	128.428	125.028	125.147	2.622	1009
7.2	116.032	113.506	113.642	2.103	1009
8	554.735	22.017	22.104	2409.6	993
9	840.459	22.718	22.671	3599.2	395
10	163.008	160.178	161.848	0.717	448
11	29.668	28.760	28.699	3.159	155
12	61.484	34.841	34.721	76.468	297
13	21.239	20.529	20.518	3.455	506

Table 3: Numerical Results for the Modified Stiffness Matrix Heuristics (GBD-2). For examples 8,9, and 12, the method failed again. In general, average convergence to global optimality of 3.18% (excluding examples 8,9 and 12).

Prob.	Best UB	(R) Sol. LB	GBD LB	O. Gap	# GBD cuts
1	31.834	27.707	27.574	14.892	179
2	24.025	22.175	22.185	7.230	677
3	18.734	17.695	17.675	5.873	649
4.1	143.828	138.053	142.609	0.855	103
4.2	128.021	126.157	127.511	0.400	29
5.1	186.441	184.448	186.411	0.016	13
5.2	123.163	120.885	121.181	1.636	699
6.1	132.205	128.212	128.156	3.114	949
6.2	120.799	116.903	116.928	3.311	449
7.1	129.662	125.028	124.991	3.707	799
7.2	117.304	113.506	113.499	3.346	449
8	35.767	22.017	22.096	61.870	1268
9	26.731	22.718	22.731	17.600	994
10	163.500	160.178	160.635	1.784	250
11	29.442	28.760	28.722	2.373	682
12	49.592	34.841	34.756	42.336	399
13	20.838	20.529	20.535	1.478	649

Table 4: Numerical Results for GBD-RENS Heuristics (GBD-3). For examples 8,9, and 12, the method failed again, but performed better over these examples. In general, average convergence to global optimality of 3.57% (excluding examples 8,9 and 12).

Prob.	Best UB	(R) Sol. LB	GBD LB	O. Gap	# GBD cuts
1	31.631	27.666	27.707	14.162	556
2	23.636	22.175	22.203	6.463	995
3	18.485	17.695	17.641	4.463	867
4.1	143.815	138.053	143.819	-0.002	129
4.2	128.021	126.157	128.021	0.0	58
5.1	186.441	184.448	186.537	-0.05	23
5.2	122.868	120.885	121.194	1.381	927
6.1	131.546	128.212	128.449	2.411	1975
6.2	119.784	116.903	116.880	2.464	1460
7.1	128.438	125.027	125.101	2.667	3593
7.2	116.075	113.506	113.559	2.216	1928
8	33.778	22.017	22.098	52.852	1230
9	26.537	22.718	22.709	16.810	298
10	162.675	160.178	162.208	0.288	902
11	29.249	28.760	28.730	1.702	93
12	45.133	34.841	34.805	29.539	1434
13	20.765	20.529	20.548	1.054	2615

Table 5: Numerical Results for GBD with Combining methods. For examples 8,9, and 12, the method failed again, but performed better than the other three methods. In general, average convergence to global optimality of 2.80% (excluding examples 8,9 and 12).

Prob.	GBD-1	GBD-2	GBD-3	GBD-4
1	14.867	<u>13.098</u>	14.892	14.162
2	21.986	<u>7.058</u>	7.230	<u>6.463</u>
3	9.237	4.864	5.873	<u>4.463</u>
4.1	0.897	0.416	0.855	<u>-0.003</u>
4.2	0.966	0.222	0.400	<u>0.000</u>
5.1	0.261	0.150	0.016	<u>-0.051</u>
5.2	1.553	1.501	1.636	<u>1.381</u>
6.1	5.696	2.441	3.114	<u>2.411</u>
6.2	4.197	2.659	3.311	<u>2.552</u>
7.1	5.439	<u>2.622</u>	3.707	2.667
7.2	3.376	<u>2.000</u>	3.346	2.216
8	2473.6	2409.6	61.870	<u>52.852</u>
9	532.8	3599.6	17.600	<u>16.810</u>
10	1.189	0.717	1.784	<u>0.288</u>
11	4.688	3.159	2.373	<u>1.702</u>
12	96.131	76.468	42.336	<u>29.539</u>
13	4.024	3.455	1.478	<u>1.054</u>
average ¹	5.56	3.18	3.57	<u>2.81</u>

¹ excluding examples 8,9 and 12

Table 6: Comparison of the convergence (O. Gap) attained by each algorithm. The smallest gap obtained among the four algorithms is underlined. The average of the optimality gaps for each method is shown at the bottom.

Another important fact to point out, is the variation in the number of GBD cuts obtained through the different examples. In general, a number of around thousand GBD cuts is an reasonable number to consider in the algorithm. Above this number, the resolution of the master problem becomes fairly slow, and almost no further improvement in the lower bound is observed. Therefore, it is desired that the algorithm uses the best quality cuts in order to converge as early as possible. In [29] it was pointed out that the quality of the GBD cuts depends strongly in the compliance value. Thus, it is natural to expect that the better the solutions found, the less number of GBD cuts will be necessary for convergence of the GBD algorithm. Furthermore, we have chosen to compare the number of GBD-cuts, because it seems to be the most fair way to compare methods following different heuristic procedures in the context of the GBD algorithm. In fact, the best way to assess a heuristic procedure, is to determine the quality of the designs obtained by this procedure, in terms of objective value, and to count the number of candidate designs found by this heuristic. In this way, a heuristic providing for example 300 candidate designs to the GBD algorithm only could be compared in a fair way to the pure GBD algorithm, after the latter has reached 300 iterations since this can be considered as a different way of exploring this number of solutions. In this sense, heuristic procedures will most likely obtain the 300 candidate designs faster than the GBD algorithm, since the GBD algorithm needs to solve one MILP problem each time a new design is obtained. Besides that, the CPU time spent in solving the MILP is unpredictable, since it depends on the intrinsic combinatorial nature of each master problem.

Nevertheless, note that for small examples the use of any heuristic procedure may make the algorithm spend time in searching candidate designs and make the overall algorithm slow in comparison with the GBD algorithm alone (GBD-1). This situation is seen in example 4.1 (the case of four angle orientation candidates). For this example, the algorithm GBD-1 stopped right away after only 4 iterations, while the algorithm GBD-4 included 23 valid feasibility cuts. But this is nothing but the conclusion that for small problems, the use of heuristics is more likely to be unnecessary.

Another interesting remark is that none of the methods could treat satisfactory Examples 1, 8, 9 and 12. There could be many reasons for this fact. Since the algorithm has shown dependency in its performance according to the successful application of heuristics, we believe that for these examples, neither the algorithm, nor the heuristics were able to find good, or close to optimal designs. If another heuristic doing this job exists, then combined with the presented GBD algorithm, it will be able obtain the best possible estimation of global convergence gap possible for the GBD algorithm. Therefore we believe that in general, the GBD method, combined with other heuristic methods, will reach better results in terms of the quality of both the obtained solution and the ability to assess the global optimality gap in numerical examples.

11 Conclusion

We have demonstrated the combined use of continuous relaxations, large neighborhood search heuristics and global integer optimization using Generalized Benders' Decomposition (GBD) for the solution of static minimum compliance multimaterial topology optimization problems with an emphasis on layered composite structures. On basis of the statement of the original nonlinear non-convex mixed-integer optimization problem, we make reformulations allowing us to solve the problem using GBD. One of the reformulations is a convex continuous relaxation on nested form which can be solved to optimality in a reasonable amount of time using a standard nonlinear programming algorithm giving both a lower bound as well as important information about the optimal solution to the original integer problem. The solution to the continuous relaxation is used within a starting heuristic in the sense that we formulate a sub-MINLP in the variables that did not obtain integer values in the continuous solution. This sub-MINLP is solved to optimality using GBD on a reduced problem and this solution is used to generate a good GBD cut in the global overall mixed-integer program. The idea of solving the reduced sub-MINLP is that the complexity of solving this problem is (hopefully) lower than solving the overall problem and by including the information obtained from its solution and the solution of the continuous relaxation, the convergence of the overall algorithm is increased compared to not including this information. Furthermore, we use a heuristic that uses a non-convex relaxation to generate cuts that lead to good, but not optimal solutions in a short amount of time. This information is also passed up to the global problem in order to speed up convergence.

The improvements in terms of the capability to solve larger problems compared to not using these heuristics are confirmed on a set of numerical examples where most instances are solved to global

optimality within a tolerance $< 3\%$. The results illustrate the combined effect of improving the lower as well as the upper bound. It is observed that improving one bound may also lead to faster improvement of the other bound.

The improvements obtained using the presented heuristics are a contribution to the ability to solve larger discrete structural optimization problems to proven global optimality. Using information from the continuous relaxation is well suited for problems where the continuous solution contains a non-negligible amount of integer-valued variables. This is often the case in structural optimization and thereby it is possible to obtain good roundings of continuous solutions.

To the authors' knowledge, the use of the RENS heuristic within a GBD (GBD-RENS) framework has not been presented before and it is our belief that the approach may be used with success with GBD for other nonlinear mixed-integer problems as well. This question remains to be investigated further by attacking and solving broader classes of different problems from e.g. some of the standard test problems for MINLP. Furthermore the use of other heuristics that (in a cheap manner) generate good cuts for the overall GBD procedure could be interesting to pursue, especially in the realm of parallel processing where individual processes could work on different heuristics and sub-problems whose information can be passed back to the overall problem and thereby improve the overall algorithm.

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References

- [1] MUST website, 2008. <http://www.mee.aau.dk/MUST/>.
- [2] Tobias Achterberg. Constraint Integer Programming. PhD thesis, Technische Universität Berlin, 2007. <http://opus.kobv.de/tuberlin/volltexte/2007/1611/>.
- [3] S. Ahmad, B.M. Irons, and O.C. Zienkiewicz. Analysis of thick and thin shell structures by curved elements. Int. J. Numer. Methods Eng., 2:419–451, 1970.
- [4] J.F. Benders. Partitioning procedures for solving mixed-variables programming problems. Numerische Mathematik, 4(1):238–252, 1962.
- [5] M.P. Bendsøe and O. Sigmund. Material interpolation schemes in topology optimization. Archive of Applied Mechanics, 69:635–654, 1999.
- [6] M.P. Bendsøe and O. Sigmund. Topology Optimization: Theory, Methods and Applications. Springer-Verlag, 2nd edition, 2003.
- [7] Timo Berthold. RENS - Relaxation Enforced Neighborhood Search. ZIB-Report 07-28, Zuse Institute Berlin, 2007. <http://opus.kobv.de/zib/volltexte/2007/1053/>.
- [8] A.V. Cherkasov. Relaxation of Problems of Optimal Structural Design. International Journal of Solids and Structures, 31(16):2251–2280, 1994.
- [9] M.A. Duran and I. E. Grossmann. An outer-approximation algorithm for a class of mixed-integer nonlinear programs. Mathematical Programming, 36(3):307–339, 1986.
- [10] RL Fox and LA Schmit. Advances in the integrated approach to structural synthesis. Journal of Spacecraft and Rockets, 3:858–866, 1966.
- [11] A. Geoffrion. Generalized Benders decomposition. Journal of Optimization Theory and Applications, 10(4):237–260, 1972.

- [12] L.V. Gibiansky and O. Sigmund. Multiphase composites with extremal bulk modulus. Journal of the Mechanics and Physics of Solids, 48(3):461–498, 2000.
- [13] P.E. Gill, W. Murray, and M.A. Saunders. SNOPT: An SQP algorithm for large-scale constrained optimization. SIAM Journal on Optimization, 12(4):979–1006, 2002.
- [14] P.E. Gill, W. Murray, and M.A. Saunders. SNOPT: An SQP algorithm for large-scale constrained optimization. SIAM Review, 47(1):99–131, 2005.
- [15] O.K. Gupta and A. Ravindran. Branch and bound experiments in convex nonlinear integer programming. Management Science, pages 1533–1546, 1985.
- [16] Gurobi Optimization. Gurobi Optimizer online reference manual, 2009. <http://www.gurobi.com/>.
- [17] H. Habibollahzadeh and JA Bubenko. Application of decomposition techniques to short-term operation planning of hydrothermal power system. IEEE Transactions on Power Systems, 1(1):41–47, 1986.
- [18] R.T. Haftka. Simultaneous analysis and design. AIAA journal, 23(7):1099–1103, 1985.
- [19] RT Haftka and MP Kamat. Simultaneous nonlinear structural analysis and design. Computational Mechanics, 4(6):409–416, 1989.
- [20] ILOG Inc. CPLEX 12.1 User’s Manual, 2009.
- [21] R. Kohn and G. Strang. Structural design optimization, homogenization and relaxation of variational problems. Macroscopic Properties of Disordered Media, pages 131–147, 1982.
- [22] R. Kohn and G. Strang. Optimal design and relaxation of variational problems. Communications in Pure and Applied Mathematics, 39:Part I: 1–25, Part II: 139–182, Part III: 353–357, 1986.
- [23] AH Land and AG Doig. An automatic method of solving discrete programming problems. Econometrica: Journal of the Econometric Society, pages 497–520, 1960.
- [24] R. Lazimy. Extension of the generalized Benders’ decomposition. Communications in Applied Numerical Methods, 2(2), 1986.
- [25] E. Lund and J. Stegmann. On structural optimization of composite shell structures using a discrete constitutive parametrization. Wind Energy, 8(1):109–124, 2005.
- [26] KA Lurie. Optimal Design of Elastic Bodies and the Problem of Regularization. Wissenschaftliche Zeitschrift, Techn. Hochschule Leipzig, 4:339–347, 1980.
- [27] T.L. Magnanti and R.T. Wong. Accelerating Benders decomposition algorithmic enhancement and model selection criteria. Operations Research, (29):464–484, 1981.
- [28] E. Munoz. Global Optimization for Structural Design by Generalized Benders’ Decomposition. PhD thesis, Department of Mathematics, Technical University of Denmark, 2010. Chapter 3.
- [29] E. Munoz. Global Optimization for Structural Design by Generalized Benders’ Decomposition. PhD thesis, Department of Mathematics, Technical University of Denmark, 2010. Chapter 4.
- [30] E. Munoz. Global Optimization for Structural Design by Generalized Benders’ Decomposition. PhD thesis, Department of Mathematics, Technical University of Denmark, 2010. Chapter 5.
- [31] F. Noonan and RJ Giglio. Planning electric power generation: A nonlinear mixed integer model employing Benders decomposition. Management Science, pages 946–956, 1977.
- [32] S. Panda and R. Natarajan. Analysis of laminated composite shell structures by finite element method. Comp. Struct., 14(3-4):225–230, 1981.
- [33] W. Rei, J. Cordeau, M. Gendreau, and P. Soriano. Accelerating Benders decomposition by local branching. INFORMS Journal on Computing, pages 1–13, 2008.

- [34] O. Sigmund and S. Torquato. Design of materials with extreme thermal expansion using a three-phase topology optimization method. Journal of the Mechanics and Physics of Solids, 45(6):1037–1067, 1997.
- [35] J. Stegmann and E. Lund. Discrete material optimization of general composite shell structures. International Journal for Numerical Methods in Engineering, 62(14):2009–2027, 2005.
- [36] M. Stolpe and J. Stegmann. A Newton method for solving continuous multiple material minimum compliance problems. Structural and Multidisciplinary Optimization, 35(2):93–106, 2008.
- [37] M. Stolpe and K. Svanberg. An alternative interpolation scheme for minimum compliance topology optimization. Structural and Multidisciplinary Optimization, 22(2):116–124, 2001.
- [38] R.A. Stubbs and S. Mehrotra. Generating convex polynomial inequalities for mixed 0–1 programs. Journal of Global Optimization, 24(3):311–332, 2002.
- [39] K. Svanberg. On the convexity and concavity of compliances. Structural and Multidisciplinary Optimization, 7(1):42–46, 1994.
- [40] A. Wächter and L.T. Biegler. On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. Mathematical Programming, 106:25–57, 2006.

CHAPTER 7

Article 5

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Global Optimization of Structural Design Problems Including Local Failure Criteria

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Abstract

This article considers the inclusion of local failure criteria in multi-material structural design problems, stated in a non-linear mixed 0-1 formulation. Our main goal is to formulate models and methods allowing us to solve the design problem to global optimality. The chosen method is the Generalized Benders' Decomposition (GBD). The local failure criteria we consider are the maximum strain, the maximum stress, the Tsai-Hill, and the Tsai-Wu criteria. We reformulate the failure criteria, into a set of convex inequalities, forming a set of convex constraints. Including these reformulations on the design problem, we obtain a mixed 0 – 1 problem with convex continuous relaxation. We can therefore use the Generalized Benders' Decomposition and/or the Outer Approximation approaches to construct an algorithm solving the design problem to global optimality. A customized GBD algorithm for this type of problems is proposed. A numerical example for fiber angle optimization is tested and solved to global optimality by use of the proposed algorithm.

Mathematical Subject Classification (2000): 90C90, 74P05, 74P15

Keywords: Discrete Material Optimization, Global Optimization, Generalized Benders' Decomposition.

1 Introduction

We consider optimal design problems of composite laminated structures including local failure criteria. We aim to solve these problems to global optimality, and for this purpose, we use the Generalized Benders' Decomposition (GBD) method (see [7, 12]). We understand for a composite structure, any structure composed of two or more different materials, where at least two different faces can be recognized. We use the term composite laminate when the faces in the structures are made of layers, where a layer can also be composed of several faces of different materials. The use of composite laminated structures has become more and more popular in the engineering field, and specially in the design of a large set of mechanical structures. It is specially important in the type of structures where the minimization of the weight is important, for manufacturing cost minimization. However, in most of the cases, for technical reasons, it is also important

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to keep or maximize other global (or local) mechanical properties of the structure. The number of structures that are designed including composite materials has grown enormously in the last years, as a consequence of the development of new techniques for the manufacture of this type of structures. The mathematical model of composite laminated structures is done by the *Finite Element Method* (FEM). We are interested in the design of thin structures. Therefore, the FEM models we use in the numerical examples are shell element models (see [19]). With respect to the optimization modeling, we use the so called *Discrete Material Optimization* (DMO) approach (see [21]). Here, we consider a finite set of candidate materials, where each of them is specified by a distinct material properties C_1, C_2, \dots, C_{n^c} . The design domain (the subdomain of the structure, where the material choice must be done) is discretized in a set of design subdomains or design elements. In each of these design elements, exactly one material must be chosen, out of the set of available candidate materials. This means that for each candidate material and for each design element, a 0 – 1 variable is included in the optimization problem. The obtained optimization problems are non linear mixed 0 – 1 programs, which we will attack without changing their mixed 0 – 1 nature. One of the advantages of the DMO approach is the fact that when the optimization problems arising from mixed 0 – 1 problems are feasible, always posses an optimal solution, which is a global optimal solution. This is not the case in models where design variables are relaxed to the continuous space $[0, 1]$. However, the optimization techniques available are not always able to find global solutions. In general, global optimization methods require convexity of the models, in order to be able to solve them efficiently. In this article, we aim to attack minimum compliance and minimum weight problems, where an additional local failure criterion is included in the design problem. In general, local failure criteria are mathematically modeled as nonconvex functions on the optimization variables. We are interested in the failure criteria that can be reformulated as convex functions on the design variables. Previously, [22] reformulated stress and displacement constraints as linear (therefore convex) 0 – 1 constraints by the so called “Big M” reformulation, and solved a 0 – 1 topology design problem by a branch-and-bound method. We aim to include in our optimization models polynomial failure criteria, as the maximum strain, the maximum stress, and the Tsai-Hill/Tsai-Wu criteria (see [11]). Similar “Big M” reformulations allow us to represent these failure criteria as a set of convex constraints. In order to solve the optimization problems obtained, we use the Generalized Benders’ Decomposition (see [7, 12]), a well-known method for mixed integer optimization. The method is based on splitting the original mixed integer problem in two simpler optimization problems. The first problem fixes the value of the integer variables and solve an optimization problem considering only the continuous variables. The second problem is a linear mixed integer problem considering only the integer variables of the original problem. The method iterates between these two problems, until it proves that the incumbent solution is a global optimal solution, with respect to a given numerical tolerance.

The article is organized as follows: Section 2 introduces the structural optimization problem with local failure criterion that is studied along this article. Section 3, presents briefly each of the local failure criteria studied in this article, and their reformulations as linear or convex set of constraints. Section 4 shows the reformulation of the equilibrium equations as a system of linear inequalities. Section 5 states the reformulated optimization models of the minimum compliance and minimum weight with local failure criteria. Section 6 presents the GBD strategy for solving the optimization models considered in this article. The statement of the general method for solving structural problems with local failure criteria by GBD is presented in Section 7. Then, the feasibility analysis for the studied failure criteria is presented in Section 9. Section 10 describes the numerical implementation

of the method, while Section 11 presents a set of numerical examples for the considered problems. Finally, section 12 presents the conclusions, a final discussion and an outline of the future work.

2 Problem Statement

We consider a given design domain of a structure $\Omega_c \subset \mathbb{R}^2$ or $\Omega_c \subset \mathbb{R}^3$, with piece-wise differentiable boundary. After a design element discretization process, the design space consists in a set of n design elements, where in each of them, a choice between n^c (non-void) candidate materials has to be made. We suppose that each candidate material is related to a specific angle orientation with respect to the global coordinate system of the structure. The situation can be easily generalized to the case where the candidate materials set is not necessarily the same for each design element. This arrangement allows us to consider for example a material candidate that may be included in two different angles, as two different candidate materials, making the mathematical modeling of the design problem easier to understand. For each of these candidate materials, a 0-1 variable is linked to its corresponding design element. The discretization introduces $\tilde{n} = n \cdot n^c$ design variables. Therefore, a design is specified by a vector x on the design space $\{0, 1\}^{\tilde{n}}$. We will use the double index notation $x_{ij} = x((j-1) \cdot n^c + i)$, $i \in \{1, \dots, n^c\}$, $j \in \{1, \dots, n\}$, to indicate the scalar design variable of the vector x related to the design element j and to the candidate material i . However, in some cases, we will use the double index notation to indicate the complete vector in $\mathbb{R}^{\tilde{n}}$, whenever the use of the indexes simplifies the understanding of the equations involved. The design problem we study consists in finding the optimal design $x^* \in \{0, 1\}^{\tilde{n}}$ minimizing an objective function. In the context of Discrete Material Optimization (see [21]), any choice of a vector $x \in \{0, 1\}^{\tilde{n}}$ represents a particular distribution of material in the structure. The governing equations are given by the equilibrium between the given external static forces $f_l \in \mathbb{R}^d$, $l = 1, \dots, m$, and the corresponding deformation of the structure. The response of the structure to these loads is computed by the finite element method, which in the case of linear elasticity, is given by the equilibrium equations

$$K(x)u_l = f_l, \quad l = 1, \dots, m, \quad (1)$$

where the matrix $K(x) \in \mathbb{R}^{d \times d}$ is the stiffness matrix related to the design x . In the case of multi-material design problems, we impose the condition of choosing exactly one candidate material for each design element. This condition is modeled by the set of constraints

$$\sum_{i=1}^{n^c} x_{ij} = 1, \quad j = 1, \dots, n. \quad (2)$$

If we assume that condition (2) holds, then equations (1) possess a unique solution. This is a consequence of the positive definiteness of the stiffness matrix in this type of problems. It implies also the existence of the inverse of the stiffness matrix $K^{-1}(x)$. The solutions $u_l \in \mathbb{R}^d$, $l = 1, \dots, m$ of (1) are the displacements of the structure under each load condition. d represents the number of degrees of freedom in the structure induced by the finite element discretization, and m is the number of external static load conditions. The first design problem studied in this article is the worse case minimum compliance problem, formulated as

$$\begin{aligned}
& \underset{\substack{x \in \mathbb{R}^{\tilde{n}} \\ u_1, \dots, u_m \in \mathbb{R}^d}}{\text{minimize}} & \max_{1 \leq l \leq m} \{f_l^T u_l\} \\
& \text{subject to} & K(x)u_l = f_l, \quad l = 1, \dots, m, \\
& & \rho^T x \leq M, \\
& & \sum_{i=1}^{n^c} x_{ij} = 1, \quad j = 1, \dots, n, \\
& & F(x, u_l) \leq 0, \quad l = 1, \dots, m, \\
& & x \in \{0, 1\}^{\tilde{n}},
\end{aligned} \tag{3}$$

where $M > 0$ is the mass or weight limit for the design, and $\rho \in \mathbb{R}^{\tilde{n}}$ is the vector of material densities for each design variable. The function

$$\begin{aligned}
F : \mathbb{R}^{\tilde{n}} \times \mathbb{R}^d & \longrightarrow \mathbb{R}^q \\
F(x, u) & \longrightarrow (F_1((x, u), \dots, F_q(x, u))
\end{aligned}$$

represents the failure function, and $F_i : \mathbb{R}^{\tilde{n}} \times \mathbb{R}^d \longrightarrow \mathbb{R}$, $i = 1 \dots, q$ are the local components of the failure function. Thus, the constraint $F(x, u_l) \leq 0$ indicates the non failure condition for the external load f_l . Problem (3) is a non-linear mixed 0 – 1 program. Of special importance is the continuous relaxation of a reformulation of the minimum compliance problem without local failure criteria, given by

$$\begin{aligned}
& \underset{x \in \mathbb{R}^{\tilde{n}}}{\text{minimize}} & \max_{1 \leq l \leq m} \{f_l^T K(x)^{-1} f_l\} \\
& \text{subject to} & \rho^T x \leq M, \\
& & \sum_{i=1}^{n^c} x_{ij} = 1, \quad j = 1, \dots, n, \\
& & x \in [0, 1]^{\tilde{n}}.
\end{aligned} \tag{4}$$

The second design problem studied in this article is the minimum weight problem, formulated as

$$\begin{aligned}
& \underset{\substack{x \in \mathbb{R}^{\tilde{n}} \\ u_1, \dots, u_m \in \mathbb{R}^d}}{\text{minimize}} & \rho^T x \\
& \text{subject to} & K(x)u_l = f_l, \quad l = 1, \dots, m, \\
& & f_l^T u_l \leq C_l, \\
& & \sum_{i=1}^{n^c} x_{ij} = 1, \quad j = 1, \dots, n, \\
& & F(x, u_l) \leq 0, \quad l = 1, \dots, m, \\
& & x \in \{0, 1\}^{\tilde{n}},
\end{aligned} \tag{5}$$

where $C_l > 0$ is the compliance limit for the design. Problem (5) is again a non-linear mixed 0–1 program. Of special importance is the continuous relaxation of a reformulation of the minimum weight problem without local failure criteria, given by

$$\begin{aligned}
& \underset{x \in \mathbb{R}^{\tilde{n}}}{\text{minimize}} & \rho^T x \\
& \text{subject to} & f_l^T K(x)^{-1} f_l \leq C_l, \\
& & \sum_{i=1}^{n^c} x_{ij} = 1, \quad j = 1, \dots, n, \\
& & x \in [0, 1]^{\tilde{n}}.
\end{aligned} \tag{6}$$

2.1 Assumptions

- (A-1) The stiffness matrix $K(x)$ is symmetric, linear in x , and positive definite for all $x \in \{0, 1\}^{\tilde{n}}$. The matrix $K(x)$ is given by

$$K(x) = \sum_{i,j} x_{ij} K_{ij}, \quad (7)$$

where $K_{ij} \in \mathbb{R}^{d \times d}$ is the symmetric positive semi definite element stiffness matrix for the j -th design element and the i -th design candidate material. K_{ij} is given explicitly by

$$K_{ij} = B_j C'_i B_j^T, \quad (8)$$

where B_j is the local strain-displacement matrix related to the design element j , C'_i is the non null symmetric and semi-positive definite strain-stress matrix for the candidate material i in the global coordinate system. This means that

$$C'_i = \Theta_i C_i \Theta_i^T,$$

with C_i the strain-stress matrix in principal material coordinate system, and Θ_i is the rotation matrix for the candidate material i between the material coordinate system (system 1 – 2) and the global coordinate system (system $x - y$). For further details, see [11].

- (A-2) The mass limit, M satisfies $0 < M < \sum_{j=1}^n \max_i \{\rho_{ij}\}$, where $\rho_{ij} \geq 0 \forall i, j$.
- (A-3) Each of the loads f_1, \dots, f_m is non null, i.e., $f_l \in \mathbb{R}^d \setminus \{0\}, \forall l = 1, \dots, m$.
- (A-4) For all load cases, the corresponding displacements are bounded, i.e., there exists bounds $u_l^{min}, u_l^{max} \in \mathbb{R}^d$, such that $u_l^{min} \leq u_l \leq u_l^{max}$.
- (A-5) The structures are deformed under the given load conditions, in a plane stress state.
- (A-6) The failure function $F(x, u)$ is a continuous function.
- (A-7) For each design element, there is a unique set of candidate materials, composed of n^c candidate materials. This assumption sets the number of design variables of the design problem to $\tilde{n} = n \cdot n^c$. This assumption makes the modeling and notation much simpler, but in practice it is not necessary.

3 Convex Formulation of Failure Criteria

In this section, we make a brief study of the mathematical properties of some relevant failure functions, which could be used for modeling multi-material design problems. In particular, we investigate the reformulation of the maximum strain, maximum stress, Tsai-Hill and Tsai-Wu failure criteria (see [11], [23]), into convex inequalities. These reformulations allow us to formulate relevant optimization design problems, which can be solved theoretically to global optimality.

3.1 The Max Strain Criterion

The maximum strain criterion is one of the simplest existing failure criteria (for a short physical explanation, see [11]). It is given by the condition

$$\| \epsilon_{i,l}(\omega) \|_{\infty} \leq \epsilon_{\max}, \quad l = 1, \dots, m, \quad \omega \in \Omega_c, i = 1, \dots, n^c, \quad (9)$$

where $\epsilon_{i,l}(\omega) \in \mathbb{R}^3 = (\epsilon_{11,i,l}, \epsilon_{22,i,l}, \epsilon_{12,i,l})^T$ represents the strain vector related to the load condition l , if the candidate material i is chosen at $\omega \in \Omega_c$. The strain vector must be measured in the corresponding material coordinate system, since $\epsilon_{\max} \in \mathbb{R}$ is the limit value for the inf-norm of the strain tensor in the material coordinate system of the candidate material i . In general the maximum strain criterion corresponds to a non linear failure constraint in the strain. It can be easily reformulated as a linear failure criterion on the strain

$$\begin{aligned} \epsilon_{i,l}(\omega) &\leq \epsilon_{\max}, & l = 1, \dots, m, \quad \omega \in \Omega_c, i = 1, \dots, n^c, \\ -\epsilon_{i,l}(\omega) &\leq \epsilon_{\max}, & l = 1, \dots, m, \quad \omega \in \Omega_c, i = 1, \dots, n^c. \end{aligned} \quad (10)$$

The strain limits can be different for each component of the strain vector, and they could also be material dependent. In this case, the limit value $\epsilon_{\max} \in \mathbb{R}$ must be replaced by a material dependent strain limit vector $\epsilon_{\max,i} \in \mathbb{R}^3$, depending on the material chosen. In this case, the max strain failure criterion is modeled as

$$\begin{aligned} \epsilon_{i,l}(\omega) &\leq \epsilon_{\max,i}, & l = 1, \dots, m, \quad \text{if material } i \text{ is chosen at } \omega, \\ -\epsilon_{i,l}(\omega) &\leq \epsilon_{\max,i}, & l = 1, \dots, m, \quad \text{if material } i \text{ is chosen at } \omega. \end{aligned} \quad (11)$$

We can even generalize to the case where the strain limits are different in tension and in compression states. After the discretization of the design space, the non failure condition (11) is written as

$$\begin{aligned} \sum_{i=1}^{n^c} x_{ij} \epsilon_{ijl} &\leq \sum_i x_{ij} \epsilon_{\max,i}, & l = 1, \dots, m, j = 1, \dots, n, \\ -\sum_{i=1}^{n^c} x_{ij} \epsilon_{ijl} &\leq \sum_i x_{ij} \epsilon_{\max,i}, & l = 1, \dots, m, j = 1, \dots, n, \end{aligned} \quad (12)$$

where $\epsilon_{ijl} = \epsilon_{i,l}(\omega_j)$, with ω_j being an evaluation point related to the j -th design element. We are allowed to model the strains and strengths as linear sums over the set of candidate materials, because in our original design problem (3), the selection constraint force the optimization program to choose only one candidate from the candidate material set. The strain vector ϵ_{ijl} in the material coordinates system is related to the strain vector ϵ_{jl}^{xy} in the global coordinates system by the introduction of the orientation rotation matrix Θ_i (related to the candidate material i , see [11]) as

$$\epsilon_{ijl} = \Theta_i^T \epsilon_{jl}^{xy}.$$

In addition, we suppose that the strains ϵ_{jl}^{xy} , $j = 1, \dots, n$, can be represented as a linear function in the displacements (see, for example [19], pp. 88, pp. 115)

$$\begin{aligned} \epsilon_{jl}^{xy} &= B_j^T u_l \\ \epsilon_{ijl} &= \Theta_i^T B_j^T u_l, \end{aligned}$$

where $B_j \in \mathbb{R}^{d \times q}$, $j = 1, \dots, n$ are the local strain-displacement matrices, and q is the size of the strain vector. For plane stress states, $q = 3$.

As a general assumption, we consider that the strains can be represented as bilinear functions in the design variable and displacements (see, for example [19], pp. 115, pp. 230) as

$$\epsilon_{jl} = \sum_{i=1}^{n^c} x_{ij} \Theta_i B_j^T u_l. \quad (13)$$

We obtain the following representation of the maximum strain failure criterion

$$\begin{aligned} \sum_{i=1}^{n^c} x_{ij} \Theta_i B_j^T u_l &\leq \sum_{i=1}^{n^c} x_{ij} \epsilon_{\max,i}, \quad j = 1, \dots, n, \quad l = 1, \dots, m \\ - \sum_{i=1}^{n^c} x_{ij} \Theta_i B_j^T u_l &\leq \sum_{i=1}^{n^c} x_{ij} \epsilon_{\max,i}, \quad j = 1, \dots, n, \quad l = 1, \dots, m, \end{aligned} \quad (14)$$

which is a set of local bilinear constraints (at the design element level) on the design variable x and the displacement field u . The constraints in (14) are non convex, but they can be reformulated as a set of linear constraints. We introduce some extra continuous variables d_{ijl} , $\forall i, j, l$, and reformulate (14) as

$$\begin{aligned} \sum_{i=1}^{n^c} d_{ijl} &\leq \sum_{i=1}^{n^c} x_{ij} \epsilon_{\max,i}, \quad \forall j, k, \\ - \sum_{i=1}^{n^c} d_{ijl} &\leq \sum_{i=1}^{n^c} x_{ij} \epsilon_{\max,i}, \quad \forall j, k, \\ d_{ijl} &= x_{ij} \Theta_i B_j^T u_l, \quad \forall i, j, k, \end{aligned}$$

and we use the following result stated and proven in [22].

Proposition 1. *Let $b \in \mathbb{R}^d$, $u^{\min}, u^{\max} \in \mathbb{R}^d$ be given constants and let $\mathcal{M} = \{(x, u, s) \in \{0, 1\} \times \mathbb{R}^d \times \mathbb{R} : u^{\min} \leq u \leq u^{\max}\}$. Further, let the constant numbers d^{\min} and d^{\max} be given by*

$$\begin{aligned} d^{\min} &= \min_u \{b^T u | u^{\min} \leq u \leq u^{\max}\} = \sum_{i:b_i > 0} b_i u_i^{\min} + \sum_{i:b_i < 0} b_i u_i^{\max} \\ d^{\max} &= \max_u \{b^T u | u^{\min} \leq u \leq u^{\max}\} = \sum_{i:b_i > 0} b_i u_i^{\max} + \sum_{i:b_i < 0} b_i u_i^{\min}. \end{aligned}$$

Then, $(x, u, s) \in \mathcal{M}$ satisfies the non-linear equation

$$s = x b^T u,$$

if and only if (x, u, s) satisfies the following four linear inequalities:

$$\begin{aligned} x d^{\min} &\leq s \leq x d^{\max} \\ (1-x) d^{\min} &\leq b^T u - s \leq (1-x) d^{\max}. \end{aligned}$$

Applying Proposition 1 to reformulate the failure condition (3.1), to reformulate it as a set of convex inequalities, we obtain the following set of linear inequalities.

$$\begin{aligned} \sum_{i=1}^{n^c} d_{ijl} &\leq \sum_{i=1}^{n^c} x_{ij} \epsilon_{\max,i}, & \forall j, k, \\ - \sum_{i=1}^{n^c} d_{ijl} &\leq \sum_{i=1}^{n^c} x_{ij} \epsilon_{\max,i}, & \forall j, k, \\ x_{ij} d_{ijl}^{\min} &\leq d_{ijl} \leq x_{ij} d_{ijl}^{\max}, & \forall i, j, k, \\ (1-x_{ij}) d_{ijl}^{\min} &\leq \Theta_i B_j^T u_l - d_{ijl} \leq (1-x_{ij}) d_{ijl}^{\max}, & \forall i, j, k, \end{aligned} \quad (15)$$

where the constant coefficients d_{ijl}^{\min} , d_{ijl}^{\max} are given by

$$\begin{aligned} d_{ijl}^{\min} &= \min_{u_l} \{ \Theta_i B_j^T u_l | u_l^{\min} \leq u_l \leq u_l^{\max} \}, \\ d_{ijl}^{\max} &= \max_{u_l} \{ \Theta_i B_j^T u_l | u_l^{\min} \leq u_l \leq u_l^{\max} \}, \end{aligned} \quad (16)$$

and u_l^{\min} , u_l^{\max} are valid bounds for the displacement field u_l . The resulting non failure condition (15) is a set of linear inequalities in the (x, u, d) -space, and we can use it for formulating an optimization problem.

3.2 The Max Stress Criterion

The maximum stress criterion is given by the condition

$$\| \sigma_l(w) \|_{\infty} \leq \sigma_{\max}, \quad l = 1, \dots, m, \quad w \in \Omega_c, \quad (17)$$

where $\sigma_l \in \mathbb{R}^3 = (\sigma_{11,l}, \sigma_{22,l}, \sigma_{12,l})^T$ represents the stress vector related to the the load condition l . (17) is a non convex failure criteria on the stress. Here, $\sigma_{\max} \in \mathbb{R}_+$ is the upper limit value for the inf-norm of the stress tensor. We generalize to the case where different limits are imposed depending on which material is chosen (introducing the vector of maximum allowed stresses $\sigma_{\max,i} \in \mathbb{R}^3, 1 \leq i \leq n^c$). Besides, a finite element discretization is included, and the maximum stress failure criterion is represented as the condition

$$\begin{aligned} \sigma_{jl} &\leq \sum_{i=1}^{n^c} x_{ij} \sigma_{\max,i}, \quad l = 1, \dots, m, j = 1, \dots, n, \\ -\sigma_{jl} &\leq \sum_{i=1}^{n^c} x_{ij} \sigma_{\max,i}, \quad l = 1, \dots, m, j = 1, \dots, n. \end{aligned} \quad (18)$$

As a general assumption, we consider that the stresses can be represented as a bilinear function in the design variable and displacements (see, for example [19], pp. 115, pp. 230) as

$$\sigma_{jl} = \sum_i C_i x_{ij} \Theta_i B_j^T u_l, \quad (19)$$

where $C_i \in \mathbb{R}^{3 \times 3}$ is the given strain-displacement matrix for the candidate material i . We obtain the following equivalent representation

$$\begin{aligned} \sum_{i=1}^{n^c} C_i x_{ij} \Theta_i B_j^T u_l &\leq \sum_{i=1}^{n^c} x_{ij} \sigma_{\max,i}, \quad j = 1, \dots, n, \quad l = 1, \dots, m \\ - \sum_{i=1}^{n^c} C_i x_{ij} \Theta_i B_j^T u_l &\leq \sum_{i=1}^{n^c} x_{ij} \sigma_{\max,i}, \quad j = 1, \dots, n, \quad l = 1, \dots, m, \end{aligned} \quad (20)$$

which is a set of local bilinear constraints (at the design element level) on the design variable x and the displacement field u_l . The constraints in (17) are non convex, but they can be reformulated as a set of linear constraints. We introduce some extra continuous variables d_{ijl} , $\forall i, j, l$, and reformulate (20) as

$$\begin{aligned} \sum_{i=1}^{n^c} C_i d_{ijl} &\leq \sum_{i=1}^{n^c} x_{ij} \sigma_{\max,i}, \quad \forall j, k, \\ - \sum_{i=1}^{n^c} C_i d_{ijl} &\leq \sum_{i=1}^{n^c} x_{ij} \sigma_{\max,i}, \quad \forall j, k, \\ d_{ijl} &= x_{ij} \Theta_i B_j^T u_l, \quad \forall i, j, k, \end{aligned} \quad (21)$$

Applying Proposition 1 to Condition (21), we can reformulate it as a set of convex inequalities, we obtain the following set of linear inequalities.

$$\begin{aligned}
\sum_{i=1}^{n^c} C_i d_{ijl} &\leq \sum_{i=1}^{n^c} x_{ij} \sigma_{\max,i}, & \forall j, k, \\
-\sum_{i=1}^{n^c} C_i d_{ijl} &\leq \sum_{i=1}^{n^c} x_{ij} \sigma_{\max,i}, & \forall j, k, \\
x_{ij} d_{ijl}^{\min} &\leq d_{ijl} \leq x_{ij} d_{ijl}^{\max}, & \forall i, j, k, \\
(1 - x_{ij}) d_{ijl}^{\min} &\leq \Theta_i B_j^T u_l - d_{ijl} \leq (1 - x_{ij}) d_{ijl}^{\max}, & \forall i, j, k,
\end{aligned} \tag{22}$$

where the constant coefficients d_{ijl}^{\min} , d_{ijl}^{\max} are given by (16), and u_l^{\min} , u_l^{\max} are valid bounds for the displacement field u_l . The resulting non failure condition (22) is a set of linear inequalities in the (x, u, d) -space, and we can use it for formulating an optimization problem.

3.3 Reformulation of the Tsai Hill Failure Criteria

We start by formulating the Tsai-Hill failure criterion (see [11], [23]) for orthotropic materials:

$$\begin{aligned}
F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\sigma_3^2 - 2F_{12}\sigma_1\sigma_2 - 2F_{13}\sigma_1\sigma_3 \\
- 2F_{23}\sigma_2\sigma_3 + 2F_{44}\sigma_4^2 + 2F_{55}\sigma_5^2 + 2F_{66}\sigma_6^2 \leq 1,
\end{aligned} \tag{23}$$

for the failure strength parameters F_{ij} . Criterion (23) applies to each of the load conditions. Considering assumption (A-5), i.e., considering that the structure deforms in a plane stress state (this is a reasonable assumption for laminated shell structures). By taking this into account, (23) is reduced to

$$F_{11}\sigma_1^2 - 2F_{12}\sigma_1\sigma_2 + F_{22}\sigma_2^2 + F_{66}\sigma_6^2 \leq 1.$$

In particular, this failure criterion can be written as a matrix equation

$$\sigma^T \mathcal{H} \sigma \leq 1, \tag{24}$$

where \mathcal{H} and σ are given by

$$\mathcal{H} = \begin{pmatrix} F_{11} & F_{12} & 0 \\ F_{12} & F_{22} & 0 \\ 0 & 0 & F_{66} \end{pmatrix}, \quad \sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{pmatrix}. \tag{25}$$

Notice that the matrix \mathcal{H} is symmetric and positive semi-definite if $F_{22} - \frac{F_{12}^2}{F_{11}} \leq 0$. This assumption is also reasonable, and some authors, such as [13], even proposed to set $F_{12} = 0$ in their models. Therefore, we included as an additional assumption to our framework, the following statement.

(A-8) The failure coefficients F_{11}, F_{22}, F_{12} , satisfy $F_{22} - \frac{F_{12}^2}{F_{11}} \leq 0$.

Thus, in the case where \mathcal{H} positive semidefinite, this matrix can be factorized, by the Choleski decomposition, as the product

$$\mathcal{H} = \mathcal{L}^T \mathcal{L},$$

with \mathcal{L} the upper triangular matrix given by

$$\mathcal{L} = \begin{pmatrix} \sqrt{F_{11}} & \frac{F_{12}}{\sqrt{F_{11}}} & 0 \\ 0 & \sqrt{F_{22} - \frac{F_{12}^2}{F_{11}}} & 0 \\ 0 & 0 & \sqrt{F_{66}} \end{pmatrix}. \tag{26}$$

Then, it follows that the non failure condition for the Tsai-Hill criterion can be written as a quadratic function in the auxiliary variable $s \in \mathbb{R}^3$

$$s^T s \leq 1, \quad (27)$$

where $s = \mathcal{L}\sigma$. After the discretization in design elements, and the inclusion of the set of design variables x_{ij} , $i = 1, \dots, n^c$, $j = 1, \dots, n$, the vector of discrete stresses $\sigma = \{\sigma_{ijl}\} \in \mathbb{R}^{3n}$

$$\sigma_{ijl} = C_i x_{ij} \Theta_i B_j^T u_l$$

is introduced. Therefore, the discretization of the non failure condition (24), considering the load conditions f_1, \dots, f_m , becomes

$$\sum_{i=1}^{n^c} \sigma_{ijl}^T \mathcal{H} \sigma_{ijl} \leq 1, \quad j = 1, \dots, n, \quad l = 1, \dots, m.$$

Then, after the inclusion of the discretized variables $\{s_{ijl}\} \in \mathbb{R}^3$, $s_{ijl} = \mathcal{L}\sigma_{ijl}$, the condition (27) becomes

$$\sum_{i=1}^{n^c} s_{ijl}^T s_{ijl} \leq 1, \quad j = 1, \dots, n, \quad l = 1, \dots, m. \quad (28)$$

Then, by using the bilinear representation of the stresses (19), the non failure condition (28) can be explicitly related to the design variable $x = \{x_{ij}\}$

$$\begin{aligned} \sum_{i=1}^{n^c} s_{ijl}^T s_{ijl} &\leq 1, \quad j = 1, \dots, n, \quad l = 1, \dots, m, \\ s_{ijl} &= x_{ij} B_{ij}^T u_l, \quad j = 1, \dots, n, \quad i = 1, \dots, n^c, \quad l = 1, \dots, m, \end{aligned} \quad (29)$$

where $B_{ij} \in \mathbb{R}^{d \times 3}$, for all i, j , are given by

$$B_{ij} = B_j \Theta_i^T C_i^T \mathcal{L}^T. \quad (30)$$

It follows that (29) is a system of non convex functions representing the Tsai-Hill failure criterion on the (x, u, s) -space. It can be reformulated as a quadratic function of the displacement fields u_l in local coordinates

$$F_{jl}^{sh}(x, u_l) = u_l^T W_j(x) u_l \leq 1, \quad j = 1, \dots, n, \quad l = 1, \dots, m, \quad (31)$$

where $W_j(x) \in \mathbb{R}^{d \times d}$ is the failure matrix related to the j -th design element. It is given as

$$W_j(x) = \sum_{i=1}^{n^c} x_{ij} W_{ij} \quad (32)$$

and $W_{ij} \in \mathbb{R}^{d \times d}$ specifies the failure matrix for the j -th design element and the i -th candidate material

$$W_{ij} = (C_i \Theta_i B_j^T)^T \mathcal{H} C_i \Theta_i B_j^T.$$

The function $F_{jl}^{sh}(x, u_l)$ is a 3rd degree polynomial in (x, u_l) . It is a non convex inequality and therefore, methods for convex optimization will not provide satisfactory solutions in terms of guaranteeing global solutions. To fix this problem, we use Proposition 1 to

reformulate (29) as a set of convex inequalities. Namely, after the reformulation, (29) becomes

$$\begin{aligned} \sum_{i=1}^{n^c} s_{ijl}^T s_{ijl} &\leq 1, & \forall j, k, \\ x_{ij} s_{ijl}^{\min} &\leq s_{ijl} \leq x_{ij} s_{ijl}^{\max}, & \forall i, j, k, \\ (1 - x_{ij}) s_{ijl}^{\min} &\leq B_{ij}^T u_l - s_{ijl} \leq (1 - x_{ij}) s_{ijl}^{\max}, & \forall i, j, k, \end{aligned} \quad (33)$$

where the constant coefficients s_{ijl}^{\min} , s_{ijl}^{\max} are given by

$$\begin{aligned} s_{ijl}^{\min} &= \min_{u_l} \{B_{ij}^T u_l | u_l^{\min} \leq u_l \leq u_l^{\max}\}, \\ s_{ijl}^{\max} &= \max_{u_l} \{B_{ij}^T u_l | u_l^{\min} \leq u_l \leq u_l^{\max}\}, \end{aligned} \quad (34)$$

and u_l^{\min} , u_l^{\max} are valid bounds for the displacement field u_l .

3.4 Reformulation of the Tsai-Wu Failure Criteria

The Tsai-Wu Failure criterion for orthotropic materials is given by

$$\sum_{i,j=1}^6 F_i \sigma_i + F_{ij} \sigma_i \sigma_j \leq 1,$$

where F_i , F_{ij} are strength tensors of the second and forth rank, respectively (see [11]). Under a plane stress assumption, the Tsai-Wu criterion becomes

$$F_1 \sigma_1 + F_2 \sigma_2 + F_6 \sigma_6 + F_{11} \sigma_1^2 + F_{22} \sigma_2^2 + F_{66} \sigma_6^2 + 2F_{12} \sigma_1 \sigma_2 \leq 1. \quad (35)$$

In particular, this failure criterion can be written as a matrix equation

$$\sigma^T \mathcal{H} \sigma + h^T \sigma \leq 1, \quad (36)$$

where \mathcal{H} is given by (25), and h are given by

$$h = \begin{pmatrix} F_1 \\ F_2 \\ F_6 \end{pmatrix}.$$

Then, by using the stress-displacement relationship given by (19), It follows that the Tsai-Wu failure criterion can be expressed as a quadratic function of the displacement field u_l in local coordinates

$$F_{jl}^{sw}(x, u_l) = u_l^T W_j(x) u_l + w_j(x) u_l \leq 1, \quad j = 1, \dots, n, \quad l = 1, \dots, m \quad (37)$$

where $W_j(x) \in \mathbb{R}^{d \times d}$ is given by (32), and $w_j(x)$ are the vectors given by

$$w_j(x) = \sum_{i=1}^{n^c} x_{ij} h^T C_i \Theta_i B_j^T.$$

First, introducing new variables s_{ijl} , η_{ijl} , $i = 1, \dots, n^c$, $j = 1, \dots, n$, $l = 1, \dots, m$, the Tsai-Wu failure criterion (36) can be written as

$$\begin{aligned} \sum_{i=1}^{n^c} s_{ijl}^T s_{ijl} + \eta_{ijl} &\leq 1, & \forall j, l, \\ s_{ijl} &= x_{ij} B_{ij}^T u_l, & \forall i, j, k, \\ \eta_{ijl} &= x_{ij} h^T C_i \Theta_i B_j^T u_l, & \forall i, j, k. \end{aligned} \quad (38)$$

Thus, using Proposition 1, we can reformulate the Tsai-Wu failure criterion into the following set of convex inequalities:

$$\begin{aligned}
\sum_{i=1}^{n^c} s_{ijl}^T s_{ijl} + \eta_{ij} &\leq 1, & \forall j, k, \\
x_{ij} s_{ijl}^{\min} &\leq s_{ijl} &\leq x_{ij} s_{ijl}^{\max}, & \forall i, j, k, \\
(1 - x_{ij}) s_{ijl}^{\min} &\leq B_{ij}^T u_l - s_{ijl} &\leq (1 - x_{ij}) s_{ijl}^{\max}, & \forall i, j, k, \\
x_{ij} \eta_{ijl}^{\min} &\leq \eta_{ijl} &\leq x_{ij} \eta_{ijl}^{\max}, & \forall i, j, k, \\
(1 - x_{ij}) \eta_{ijl}^{\min} &\leq h^T C_i \Theta_i B_j^T u_l - \eta_{ijl} &\leq (1 - x_{ij}) \eta_{ijl}^{\max} & \forall i, j, k,
\end{aligned} \tag{39}$$

where the constant coefficients s_{ijl}^{\min} , s_{ijl}^{\max} are given by (34), while η_{ijl}^{\min} , η_{ijl}^{\max} are given by

$$\begin{aligned}
\eta_{ijl}^{\min} &= \min_{u_l} \{ h^T C_i B_j^T u_l | u_l^{\min} \leq u_l \leq u_l^{\max} \}, & \forall i, j, k, \\
\eta_{ijl}^{\max} &= \max_{u_l} \{ h^T C_i B_j^T u_l | u_l^{\min} \leq u_l \leq u_l^{\max} \}, & \forall i, j, k.
\end{aligned} \tag{40}$$

4 Reformulation of the Equilibrium Equations

The equilibrium equations (1), with stiffness matrix $K(x)$ given by (7), are bilinear equality constraints in the design variable x and displacement fields u_l , $l = 1, \dots, m$. They are therefore, a set of non convex constraints. This situation can be solved by using the linear reformulation based on Proposition 1. First we introduce the additional variables $r_l = \{r_{ijl}\} \in \mathbb{R}^{\tilde{n}}$, for $l = 1, \dots, m$, given by

$$r_{ijl} = x_{ij} C_i \Theta_i B_j^T u_l,$$

so the equilibrium equations can be rewritten as the system

$$\begin{aligned}
Br_l &= f_l, & l = 1, \dots, m, \\
r_l &= \{r_{ijl}\}_{ij} = \{x_{ij} C_i \Theta_i B_j^T u_l\}_{ij}, & l = 1, \dots, m.
\end{aligned} \tag{41}$$

Here, the matrix $B \in \mathbb{R}^{d \times nq}$ given by $B = [B_1, \dots, B_n]$, with $B_j \in \mathbb{R}^{d \times q}$ the strain-displacement matrix for the finite element j . Equations (41) represent still a non convex system of equations. By using Proposition 1, we expect to obtain a linear system of inequalities. This reformulation was already presented in [22] for discrete topology optimization problems. By doing this, the equilibrium equations (1) are reformulated as

$$\begin{aligned}
Br_l &= f_l & \forall l, \\
-x_{ij} r_{ijl}^{\max} + r_{ijl} &\leq 0, & \forall i, j, k, \\
x_{ij} r_{ijl}^{\min} - r_{ijl} &\leq 0, & \forall i, j, k, \\
-C_i \Theta_i B_j^T u_l - x_{ij} r_{ijl}^{\min} + r_{ijl} &\leq -r_{ijl}^{\min}, & \forall i, j, k, \\
C_i \Theta_i B_j^T u_l + x_{ij} r_{ijl}^{\max} - r_{ijl} &\leq r_{ijl}^{\max}, & \forall i, j, k,
\end{aligned} \tag{42}$$

where r_{ijl}^{\min} , r_{ijl}^{\max} are given by

$$\begin{aligned}
r_{ijl}^{\min} &= \min_{u_l} \{ C_i \Theta_i B_j^T u_l | u_l^{\min} \leq u_l \leq u_l^{\max} \}, & \forall i, j, k, \\
r_{ijl}^{\max} &= \max_{u_l} \{ C_i \Theta_i B_j^T u_l | u_l^{\min} \leq u_l \leq u_l^{\max} \}, & \forall i, j, k.
\end{aligned} \tag{43}$$

The reformulated equilibrium equations (42) is a linear 0 – 1 system. For numerical implementations where linearity is desired, it can replace the original equilibrium equations (1). For notation simplicity, the linear reformulated version of the equilibrium equations (42) will be displayed in a condensed matrix form

$$\begin{aligned} Br_l &= f_l & \forall l, \\ A(r^{\min}, r^{\max})(x, u_l, r)^T &\leq 0 & \forall l. \end{aligned}$$

5 Reformulation of the Design Problem

In this section, we present reformulations of the minimum compliance problem (3) and the minimum weight problem (5), considering different local failure criteria, namely, the maximum strain, the maximum stress, the Tsai-Hill and the Tsai-Wu criteria.

5.1 Minimum Compliance, Max Strain Criterion

The minimum compliance problem (3), considering the maximum strain criterion (9), is reformulated as a linear mixed 0 – 1 problem as

$$\begin{aligned} &\underset{\substack{x, d, r \\ u_1, \dots, u_m}}{\text{minimize}} && \max_{1 \leq l \leq m} \{f_l^T u_l\} \\ &\text{subject to} && Br_l = f_l, && l = 1, \dots, m, \\ & && A(r^{\min}, r^{\max})(x, u_l, r)^T \leq 0 && l = 1, \dots, m, \\ & && \rho^T x \leq M, \\ & && \sum_{i=1}^{n^c} d_{ijl} \leq \sum_{i=1}^{n^c} x_{ij} \epsilon_{\max, i}, && \forall j, k, \\ & && - \sum_{i=1}^{n^c} d_{ijl} \leq \sum_{i=1}^{n^c} x_{ij} \epsilon_{\max, i}, && \forall j, k, \\ & && x_{ij} d_{jl}^{\min} \leq d_{ijl} \leq x_{ij} d_{jl}^{\max}, && \forall i, j, k, \\ & && (1 - x_{ij}) d_{jl}^{\min} \leq \Theta_i B_j^T u_l - d_{ijl}, && \forall i, j, k, \\ & && \Theta_i B_j^T u_l - d_{ijl} \leq (1 - x_{ij}) d_{jl}^{\max}, && \forall i, j, k, \\ & && \sum_{i=1}^{n^c} x_{ij} = 1, && \forall j, \\ & && x \in \{0, 1\}^{\tilde{n}}, \end{aligned} \tag{44}$$

with d_{jl}^{\min} , d_{jl}^{\max} given by (16). Because of its linear 0 – 1 integer nature, it can be attacked by any method for linear mixed integer optimization, and be solved directly by some commercial software. It could also be attacked by the Benders' Decomposition method (see [3]). The draw-back of this formulation is the inclusion of a large number of variables (r and d), and the large number of constraints. This problem can also be reformulated as a problem with convex continuous relaxation, considering the maximum strain criterion (14), reformulated to a convex system of constraints (15), and the reformulation of the

equilibrium equations (1), given by the equations (42), as

$$\begin{aligned}
& \underset{\substack{x, r \\ u_1, \dots, u_m}}{\text{minimize}} && \max_{1 \leq l \leq m} \{f_l^T K(x)^{-1} f_l\} \\
& \text{subject to} && Br_l = f_l, && l = 1, \dots, m, \\
& && A(r^{\min}, r^{\max})(x, u_l, r_l)^T \leq 0 && l = 1, \dots, m, \\
& && \rho^T x \leq M, \\
& && \sum_{i=1}^{n^c} d_{ijl} \leq \sum_{i=1}^{n^c} x_{ij} \epsilon_{\max, i}, && \forall j, k, \\
& && - \sum_{i=1}^{n^c} d_{ijl} \leq \sum_{i=1}^{n^c} x_{ij} \epsilon_{\max, i}, && \forall j, k, \\
& && x_{ij} d_{ijl}^{\min} \leq d_{ijl} \leq x_{ij} d_{ijl}^{\max}, && \forall i, j, k, \\
& && (1 - x_{ij}) d_{ijl}^{\min} \leq \Theta_i B_j^T u_l - d_{ijl} \leq (1 - x_{ij}) d_{ijl}^{\max}, && \forall i, j, k, \\
& && \sum_{i=1}^{n^c} x_{ij} = 1, && j = 1, \dots, n, \\
& && x \in \{0, 1\}^{\tilde{n}},
\end{aligned} \tag{45}$$

where $r^{\min} = \{r_{ijl}^{\min}\}$ and $r^{\max} = \{r_{ijl}^{\max}\}$ are as in (43). Problem (45) is a non linear mixed 0 – 1 problem having a convex continuous relaxation. Therefore, it can be attacked by the proposed GBD method, or any method for linear mixed 0 – 1 optimization. The drawback of this formulation is the inclusion of the extra variables r , and the large number of constraints.

5.2 Minimum Compliance, Max Stress Criterion

The minimum compliance problem (3), considering the maximum stress criterion (17), is reformulated as a linear mixed 0 – 1 problem as

$$\begin{aligned}
& \underset{\substack{x, d, r \\ u_1, \dots, u_m}}{\text{minimize}} && \max_{1 \leq l \leq m} \{f_l^T u_l\} \\
& \text{subject to} && Br_l = f_l, && l = 1, \dots, m, \\
& && A(r^{\min}, r^{\max})(x, u_l, r)^T \leq 0 && l = 1, \dots, m, \\
& && \rho^T x \leq M, \\
& && \sum_{i=1}^{n^c} C_i d_{ijl} \leq \sum_{i=1}^{n^c} x_{ij} \sigma_{\max, i}, && \forall j, k, \\
& && - \sum_{i=1}^{n^c} C_i d_{ijl} \leq \sum_{i=1}^{n^c} x_{ij} \sigma_{\max, i}, && \forall j, k, \\
& && x_{ij} d_{ijl}^{\min} \leq d_{ijl} \leq x_{ij} d_{ijl}^{\max}, && \forall i, j, k, \\
& && (1 - x_{ij}) d_{ijl}^{\min} \leq \Theta_i B_j^T u_l - d_{ijl}, && \forall i, j, k, \\
& && \Theta_i B_j^T u_l - d_{ijl} \leq (1 - x_{ij}) d_{ijl}^{\max}, && \forall i, j, k, \\
& && \sum_{i=1}^{n^c} x_{ij} = 1, && \forall j, \\
& && x \in \{0, 1\}^{\tilde{n}},
\end{aligned} \tag{46}$$

with d_{ijl}^{\min} , d_{ijl}^{\max} given by (16). Because of its linear 0 – 1 integer nature, it can be attacked by any method for linear mixed integer optimization, and be solved directly by some commercial software. It could also be attacked by the Benders' Decomposition method (see [3]). The draw-back of this formulation is, as in the case of the max strain criteria, the inclusion of a large number of variables (r and d), and the large number of constraints. This problem can also be reformulated as a problem with convex continuous relaxation. Such a formulation is given by

$$\begin{aligned}
& \underset{\substack{x, d, r \\ u_1, \dots, u_m}}{\text{minimize}} && \max_{1 \leq l \leq m} \{f_l^T K(x)^{-1} f_l\} \\
& \text{subject to} && Br_l = f_l, && l = 1, \dots, m, \\
& && A(r^{\min}, r^{\max})(x, u_l, r_l)^T \leq 0 && l = 1, \dots, m, \\
& && \rho^T x \leq M, \\
& && \sum_{i=1}^{n^c} C_i d_{ijl} \leq \sum_{i=1}^{n^c} x_{ij} \sigma_{\max, i}, && \forall j, l, \\
& && - \sum_{i=1}^{n^c} C_i d_{ijl} \leq \sum_{i=1}^{n^c} x_{ij} \sigma_{\max, i}, && \forall j, l, \\
& && x_{ij} d_{ijl}^{\min} \leq d_{ijl} \leq x_{ij} d_{ijl}^{\max}, && \forall i, j, k, \\
& && (1 - x_{ij}) d_{ijl}^{\min} \leq \Theta_i B_j^T u_l - d_{ijl}, && \forall i, j, k, \\
& && \Theta_i B_j^T u_l - d_{ijl} \leq (1 - x_{ij}) d_{ijl}^{\max}, && \forall i, j, k, \\
& && \sum_{i=1}^{n^c} x_{ij} = 1, && \forall j, \\
& && x \in \{0, 1\}^{\tilde{n}},
\end{aligned} \tag{47}$$

In this last reformulation, the compliance is represented as a function on the design variables (see [4]). Problem (47) is a non linear mixed integer optimization problem with convex continuous relaxation. This problem can be attacked by any method for non linear mixed integer optimization.

5.3 Minimum Compliance, Tsai-Hill Criterion

The minimum compliance problem (3) considering the Tsai-Hill failure criterion (24), is reformulated as a problem with convex continuous relaxation, by using the convex formulation of the Tsai-Hill criterion, represented by equations (33) and (34). The existence of a convex continuous relaxation of the Tsai-Hill criterion is justified on the assumption (A-7), stated in section 3.3: The failure coefficients F_{11}, F_{22}, F_{12} satisfy $F_{22} - \frac{F_{12}}{F_{11}} \leq 0$. This reformulation is given by

$$\begin{aligned}
& \underset{\substack{x, s \\ u_1, \dots, u_m}}{\text{minimize}} && \max_{1 \leq l \leq m} \{f_l^T K(x)^{-1} f_l\} \\
& \text{subject to} && Br_l = f_l, && l = 1, \dots, m, \\
& && A(r^{\min}, r^{\max})(x, u_l, r)^T \leq 0 && l = 1, \dots, m, \\
& && \rho^T x \leq M, \\
& && \sum_{i=1}^{n^c} s_{ijl}^T s_{ijl} \leq 1, && \forall j, l \\
& && x_{ij} s_{ijl}^{\min} - s_{ijl} \leq 0 && \forall i, j, k, \\
& && s_{ijl} - x_{ij} s_{ijl}^{\max} \leq 0, && \forall i, j, k, \\
& && -x_{ij} s_{ijl}^{\min} - B_{ij}^T u_l + s_{ijl} \leq -s_{ijl}^{\min}, && \forall i, j, k, \\
& && B_{ij}^T u_l + x_{ij} s_{ijl}^{\max} - s_{ijl} \leq s_{ijl}^{\max}, && \forall i, j, k, \\
& && \sum_{i=1}^{n^c} x_{ij} = 1, && \forall j, \\
& && x \in \{0, 1\}^{\tilde{n}},
\end{aligned} \tag{48}$$

with $s_{ijl}^{\min}, s_{ijl}^{\max}$ given by (34). Problem (48) is a non linear mixed 0 – 1 optimization problem with convex continuous relaxation. We can attack this problem using for example the GBD method.

5.4 Minimum Compliance, Tsai-Wu Criterion

The minimum compliance problem (3) considering the Tsai-Wu failure criterion (35), is reformulated as a problem with convex continuous relaxation, by using the convex formulation of the Tsai-Wu criterion, represented by equations (39) and (40). The existence of a convex continuous relaxation of the Tsai-Wu criterion is justified on the assumption (A-7), stated in section 3.3: The failure coefficients F_{11}, F_{22}, F_{12} satisfy $F_{22} - \frac{F_{12}}{F_{11}} \leq 0$. This reformulation is given by

$$\begin{aligned}
& \underset{\substack{x, s, \eta \\ u_1, \dots, u_m}}{\text{minimize}} & & \max_{1 \leq l \leq m} \{f_l^T K(x)^{-1} f_l\} \\
& \text{subject to} & & Br_l = f_l, & l = 1, \dots, m, \\
& & & A(r^{\min}, r^{\max})(x, u_l, r)^T \leq 0 & l = 1, \dots, m, \\
& & & \rho^T x \leq M, \\
& & & \sum_{i=1}^{n^c} s_{ijl}^T s_{ijl} + \eta_{ij} \leq 1, & \forall j, l, \\
& & & s_{ijl} - x_{ij} s_{ijl}^{\max} \leq 0, & \forall i, j, k, \\
& & & -s_{ijl} + x_{ij} s_{ijl}^{\min} \leq 0, & \forall i, j, k, \\
& & & B_{ij}^T u_l - s_{ijl} + x_{ij} s_{ijl}^{\max} \leq s_{ijl}^{\max}, & \forall i, j, k, \\
& & & -B_{ij}^T u_l + s_{ijl} - x_{ij} s_{ijl}^{\min} \leq -s_{ijl}^{\min}, & \forall i, j, k, \\
& & & \eta_{ijl} - x_{ij} \eta_{ijl}^{\max} \leq 0, & \forall i, j, k, \\
& & & -\eta_{ijl} + x_{ij} \eta_{ijl}^{\min} \leq 0, & \forall i, j, k, \\
& & & h^T C_i \Theta_i B_j^T u_l - \eta_{ijl} + x_{ij} \eta_{ijl}^{\max} \leq \eta_{ijl}^{\max}, & \forall i, j, k, \\
& & & -h^T C_i \Theta_i B_j^T u_l + \eta_{ijl} - x_{ij} \eta_{ijl}^{\min} \leq -\eta_{ijl}^{\min}, & \forall i, j, k, \\
& & & \sum_{i=1}^{n^c} x_{ij} = 1, & \forall j, \\
& & & x \in \{0, 1\}^{\tilde{n}},
\end{aligned} \tag{49}$$

with $s_{ijl}^{\min}, s_{ijl}^{\max}$ given by (34), and $\eta_{ijl}^{\min}, \eta_{ijl}^{\max}$ given by (40). Problem (49) is a non linear mixed integer optimization problem with convex continuous relaxation. We can attack this problem using for example the GBD method.

5.5 Reformulation of the Minimum Weight Problem

In general, for the minimum weight problem with local failure criteria, the corresponding reformulation for the Max Strain, Max Stress, Tsai-Hill, Tsai-Wu criteria are analogous, and can be obtained directly from the corresponding minimum compliance formulation here presented. As an example we state the minimum weight problem with compliance constraint, and maximum stress local criterion.

$$\begin{aligned}
& \underset{\substack{x, d, r \\ u_1, \dots, u_m}}{\text{minimize}} & & \rho^T x \\
& \text{subject to} & & Br_l = f_l, & l = 1, \dots, m, \\
& & & A(r^{\min}, r^{\max})(x, u_l, r_l)^T \leq 0 & l = 1, \dots, m, \\
& & & f_l^T K(x)^{-1} f_l \leq C, & l = 1, \dots, m, \\
& & & \sum_{i=1}^{n^c} C_i x_{ij} \Theta_i B_j^T u_l \leq \sum_{i=1}^{n^c} x_{ij} \sigma_{\max, i}, & j = 1, \dots, n, \quad l = 1, \dots, m, \\
& & & -\sum_{i=1}^{n^c} C_i x_{ij} \Theta_i B_j^T u_l \leq \sum_{i=1}^{n^c} x_{ij} \sigma_{\max, i}, & j = 1, \dots, n, \quad l = 1, \dots, m, \\
& & & \sum_{i=1}^{n^c} x_{ij} = 1, & j = 1, \dots, n, \\
& & & x \in \{0, 1\}^{\tilde{n}}.
\end{aligned} \tag{50}$$

The reformulation of the minimum weight problem considering the other local failure criteria are straightforward. Therefore we will not state them explicitly here.

6 Resolution By Generalized Benders' Decomposition

In this section, we show an approach to attack the formulated problems by using the Generalized Benders' Decomposition (GBD, see [7] and [12]). GBD is a well known method for mixed integer optimization and it obtains, if convergence occurs, global optimal solutions. The method is based on the projection theory, i.e., it is based on separating the decision space in two sets of variables, namely the set integer variables and the set of continuous variables. The GBD method solves two sequences of optimization problems, each of them considering only one of the two set of variables. For details about the GBD method, refer to the articles cited above.

Problems (47), (48), and (49) are non linear mixed integer problems with convex continuous relaxation. As it was proven in [14], the GBD method ([7], [12]) applied to the minimum compliance and minimum weight problems, converges to a global optimum, as long as the corresponding continuous relaxation is convex. In the case of the formulations including local failure criteria stated in this article, the generalization of those results is expected and assumed.

6.1 Generalized Benders' Decomposition Algorithm

In [14], it was proven that the presented Generalized Benders' Decomposition algorithm applied to the minimum compliance/weight problem (without local failure criteria) converges into a global optimal solution, in a finite number of iterations. This result is important and we will use it to extend it to the case of inclusion of local failure criteria.

In particular, the results in [14] imply that if the stiffness matrix $K(x)$ is given by (7), then the GBD method applied to the minimum compliance problem

$$\begin{aligned} & \underset{x \in \mathbb{R}^{\tilde{n}}, u \in \mathbb{R}^d}{\text{minimize}} && c = f^T u \\ & \text{subject to} && K(x)u = f, \\ & && \rho^T x \leq M, \\ & && \sum_{i=1}^{n^c} x_{ij} = 1, \quad \forall j, \\ & && x \in \{0, 1\}^n. \end{aligned} \tag{51}$$

converges in a finite number of iterations to a global optimal design. Problem (51) corresponds to the problem (3), where the failure criterion has been removed. In ([15]), the GBD method was applied to (51).

From [14], [16], the GBD method applied to the problem (51) supposes the inclusion of two sequences of simpler optimization problems. The first one is the sequence of the so called *subproblems*, considering the displacement field u (a continuous variable in \mathbb{R}^d). The second one is the sequence of *master problems*, considering the design variable x (a variable in $\{0, 1\}^{\tilde{n}}$).

The subproblem corresponds to the problem (3), for which, the variable x is fixed to a given design $x = x^k \in [0, 1]^{\tilde{n}}$, so the optimization problem only takes into consideration the displacement field u

$$\begin{aligned} & \underset{u_l \in \mathbb{R}^d}{\text{minimize}} && c = \max_l \{f^T u_l\} \\ & \text{subject to} && K(x^k)u_l = f_l, \quad l = 1, \dots, m. \end{aligned} \tag{52}$$

Problem (52) simply corresponds to solve the equilibrium equations (1) by $u_l^k = K(x^k)^{-1} f_l$, and evaluate the compliance related to the design x^k by $c(x^k) = \max_l \{f^T u_l^k\}$. Subproblem (52) is also used to obtain optimal Lagrange multipliers related to the equilibrium equations. In [14] it was proven that it was necessary a reformulation of the subproblem in order to ensure the existence of these optimal Lagrange multipliers. With this information, it is possible to generate additional linear constraints $l^*(x^k, \eta^k, u^k)$, called *optimality cuts*. In the case of the problem (51), the optimality cuts l^* represent a linear approximation of the compliance function as a function of x , at x^k (see [14]). The master problem is defined almost exactly as it was defined in [14], and we repeat its description and notation used, adapted to the problem (3). The master problem for iteration N corresponds to the following linear mixed 0 – 1 problem.

$$\begin{aligned}
& \underset{x \in \mathbb{R}^{\tilde{n}}, y \in \mathbb{R}}{\text{minimize}} && y \\
& \text{subject to} && l_c^*(x, u_l^k, \nu_l^k) \leq y, \quad \forall k = 1, \dots, N, \quad l = 1 \dots, m, \\
& && \rho^T x \leq M \\
& && \sum_{i=1}^{n^c} x_{ij} = 1, \quad \forall j = 1, \dots, n, \\
& && x \in \{0, 1\}^{\tilde{n}},
\end{aligned} \tag{53}$$

where l_c^* is a function defined as

$$\begin{aligned}
l_c^*(x, u_l^k, \nu_l^k) &= f_l^T u_l^k + \nu_l^{kT} u_l^k [x^k - x], \\
\nu_l^k &= \left(u_l^{kT} K_{11} \quad u_l^{kT} K_{12} \quad \dots \quad u_l^{kT} K_{n^c n} \right)^T,
\end{aligned} \tag{54}$$

where u_l^k is given by the subproblem (52), and x^k is the solution of the k -th relaxed master problem.

The following notation rule is identical to the ones stated in [16]. We repeat it almost exactly, since it defines the notation used in this article.

Notation 1. *In the definition of the function l_c^* given by (54), a special notation is used. This notation will be used throughout the article. The expression*

$$\nu^{kT} u^k [x^k - x] = \nu^{kT} u^k x^k - \nu^{kT} u^k x,$$

should be understood in the following way. Each product of three terms

$$\nu^T u x, \text{ with } \nu \in \mathbb{R}^{\tilde{n}d}, u \in \mathbb{R}^d, x \in \mathbb{R}^{\tilde{n}},$$

is interpreted as

$$\nu^T u x = \sum_{j=1}^d \sum_{k=1}^{\tilde{n}} \nu(d[k-1] + j) u(j) x(k).$$

In particular, using the expression of ν^k in (54), the $\nu^{kT} u^k x^k$ terms are interpreted throughout the article as

$$\nu^{kT} u^k x^k = \sum_{i=1}^{n^c} \sum_{j=1}^n x_{ij}^k u^{kT} K_{ij} u^k.$$

The main purpose here is to extend the GBD method to problems of the type of (3). We show how to do this in the next section.

6.2 A Strategy for Applying Generalized Benders' Decomposition to Problems with Local Failure Criteria

In this subsection, we introduce the general approach to solve problems of the type of (3), in such a way, that the solutions obtained are still global optimal solutions. The main strategy is to consider the local failure functions $F(x, u_l)$ on the subproblem, and from this, obtain a set of valid feasibility constraints on the design space x . The advantage of this approach is that it can basically consider the Generalized Benders' Decomposition (GBD) algorithm presented ([15]) and modify only the subproblem, when including the local failure functions. In the new approach, we will consider a given load l_0 , and the subproblem

$$\begin{aligned} \underset{u_{l_0} \in \mathbb{R}^d}{\text{minimize}} \quad & c = \{0^T u_{l_0}\} \\ \text{subject to} \quad & K(x^k)u_{l_0} = f_{l_0}, \\ & F(x^k, u_{l_0}) \leq 0. \end{aligned} \tag{55}$$

The subproblem (55) will be called the *equilibrium-local failure feasibility system*. The idea is to obtain valid feasibility constraints for the local failure function through (55).

To do so, the problem (55) must be reformulated as a convex feasibility problem. In section 3, the convex reformulation of the failure functions was presented. These reformulations imply the existence of a convex version of the feasibility problem (55). It is given by

$$\begin{aligned} \underset{u_{l_0}, r, s}{\text{minimize}} \quad & c = \{0^T u_{l_0} + 0^T r + 0^T s\} \\ \text{subject to} \quad & Br_{l_0} = f_{l_0}, \\ & A(r^{\min}, r^{\max})(x^k, u_{l_0}, r)^T \leq 0 \\ & F(x^k, u_{l_0}, s) \leq 0, \end{aligned} \tag{56}$$

where $F(x^k, u_{l_0}, s)$ is a convex reformulation of the nonconvex failure function $F(x^k, u_{l_0})$. In addition, the master problem has exactly the same appearance as it has when no failure criteria is considered (i.e., the master problem given by (53)), with the only difference that now, feasibility cuts $l_*^{cr} \leq 0$ related to the feasibility system (56) are added to the master problem, whenever a design x^k is infeasible for the equilibrium and the considered failure criterion.

For each design x^k obtained after solving a relaxed master problem, one GBD cut and eventually one or more feasibility cuts are included.

1. First, we include the corresponding U -level set GBD cut

$$-\{\nu_{l,U_c}^k\}^T \{u_{l,U_c}^k\}x - y \leq -2f_l^T \{u_{l,U_c}^k\}$$

(i.e., the one related to the design x_U^k).

2. Then, if the design is not feasible for the local failure criterion, a feasibility cut

$$G_q(x) \leq 0, \quad q = 1, \dots, Q,$$

preventing x^k to be feasible in the next iterations are included.

7 Statement of the Method

In this section, we present the Generalized Benders' Decomposition (GBD) method to solve the multiple load minimum compliance problem (3). The assumptions considered are the same as in the previous sections, i.e. assumptions (A-1) – (A-8).

Algorithm 1: Generalized Benders' Decomposition for multiple load minimum compliance problem (3) with local failure criteria

1. [1A] (Optional Step) Run any heuristic procedure (if available) to obtain 0–1 designs for the optimal design problem. If P initial designs were found, number them in an arbitrary order x^1, \dots, x^P . If among these P designs found there are feasible designs for the optimization problem, set U as the objective of the incumbent solution. Call this incumbent x^* .

[1B] (if [1A] is not executed) Set $P = 1$, , the upper bound $U = +\infty$, Find an initial design $x^1 \in [0, 1]^{\tilde{n}}$ satisfying the mass constraint $\rho^T x^1 \leq M$, and the material selection constraints $\sum_{i=1}^{n^c} x_{ij} = 1, j = 1, \dots, n$.

2. Set $Q = 0$, the lower bound $y^* = -\infty$ and the convergence tolerance $\epsilon \geq 0$. Compute the displacement fields $u^k = u_l^k, \dots, u_m^k, u_l^k = K(x^k)^{-1} f_l$, the compliances $c_l^k, \dots, c_m^k, c_l^k = f_l^T u_l^k$, and the Lagrange multiplier vectors

$$\nu_l^k = ((K_{11} u_l^k)^T (K_{12} u_l^k)^T \dots (K_{n^c n} u_l^k)^T)^T,$$

for $l = 1, \dots, m, k = 1, \dots, P$. Notice that the initial design x^1 does not need to be a 0–1 design, and it does not need to satisfy the local failure criterion $F(x^1, u_l) \leq 0, l = 1, \dots, m$.

3. Solve the first relaxed master problem

$$\begin{aligned} & \underset{x \in \mathbb{R}^{\tilde{n}}, y \in \mathbb{R}}{\text{minimize}} && y \\ & \text{subject to} && -\nu_l^{kT} u_l^k x - y \leq -2f_l^T u_l^k, \quad l = 1, \dots, m, \quad k = 1, \dots, P, \\ & && \rho^T x \leq M, \\ & && \sum_{i=1}^{n^c} x_{ij} = 1, \quad j = 1, \dots, n \\ & && x \in \{0, 1\}^{\tilde{n}}, \end{aligned} \tag{M1}$$

by any solver for linear-mixed integer programming. If the problem is infeasible, stop and exit. The whole design problem is infeasible. Otherwise, if the problem is feasible, it necessarily has at least one optimal solution. This comes from the fact that the functions

$$h_l^k : \{0, 1\}^n \longrightarrow \mathbb{R}, \quad h_l^k(x) = -\nu_l^{kT} u_l^k x, \quad \forall k, l,$$

are bounded from below. Denote the solution of (M1) found by (x^{P+1}, y^{P+1}) . Set $P \rightarrow P + 1$.

4. Solve the equilibrium equations $K(x^P) u_l^P = f_l$, for $l = 1, \dots, m$, and compute the compliances $c_l^P = f_l^T u_l^P$, for $l = 1, \dots, m$.
5. For each $l = 1, \dots, m$, evaluate the equilibrium-local failure feasibility subproblem:

$$\begin{aligned} & \underset{u_1, \dots, u_m \in \mathbb{R}^d}{\text{minimize}} && \sum_{l=1}^m 0^T u_l \\ & \text{subject to} && K(x^P) u_l = f_l, \quad l = 1, \dots, m, \\ & && F(x^P, u_l) \leq 0. \end{aligned}$$

If the subproblem (57) is infeasible, compute one or more valid feasibility cuts preventing the design x^P from being feasible in the next master problem (this could be done, for example, by building GBD feasibility cuts after treating the reformulation of the feasibility system as a convex problem). Denote $G_{Q+1}(x) \leq d_{Q+1}, \dots, G_{Q+p}(x) \leq d_{Q+p}$ the found feasibility cuts. Set $Q \rightarrow Q + p$. Compute then the Lagrange multiplier vectors

$$\nu_l^P = ((K_{11}u_l^P)^T (K_{12}u_l^P)^T \dots (K_{n^c n}u_l^P)^T)^T, \quad \forall l = 1, \dots, m.$$

6. If (57) is feasible, and $\max_{1 \leq l \leq m} \{c_l^P\} < U$, then set $U = \max_{1 \leq l \leq m} \{c_l^P\}$, $x^* = x^P$, and update all designs, displacements and Lagrange multipliers to the level set of U (as it is explained in [17]), obtaining x_U^k , $u_{l,U}^k$, and $\nu_{l,U}^k$, $k = 1, \dots, P$, $l = 1, \dots, m$.
If (57) is feasible, and $\max_{1 \leq l \leq m} \{c_l^P\} \geq U$, compute the level set design x_U^P (as it is explained in [17]), the corresponding displacements $u_{l,U}^P$, and Lagrange multipliers $\nu_{l,U}^P$, $l = 1, \dots, m$.
7. If $U - y^* \leq \epsilon$, then stop. The optimal design found is x^* , with optimal value U . Otherwise, set $P \rightarrow P + 1$, and continue to step 8.
8. Solve the relaxed master problem

$$\begin{aligned} & \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\ & \text{subject to} && -\nu_l^{kT} u_l^k x - y \leq -2f_l^T u_l^k, & k = 1, \dots, P, \quad l = 1, \dots, m, \\ & && -\{\nu_{l,U_c}^k\}^T \{u_{l,U_c}^k\} x - y \leq -2f_l^T \{u_{l,U_c}^k\}, & k = 1, \dots, P, \quad l = 1, \dots, m, \\ & && G_q(x) \leq 0, & q = 1, \dots, Q, \\ & && \rho^T x \leq M, \\ & && \sum_{i=1}^{n^c} x_{ij} = 1, & j = 1, \dots, n, \\ & && x \in \{0, 1\}^{\tilde{n}}, \end{aligned}$$

by any solver for linear-mixed 0 – 1 programming. If the relaxed master problem is infeasible, stop and exit. If a feasible design has been found, it is an optimal solution for the design problem. Otherwise, the design problem is infeasible. If the relaxed master problem is feasible and an optimal solution is found, denote the solution of this program by (x^{P+1}, y^{P+1}) . Set $P \rightarrow P + 1$. Return to step 4.

Remark 1. *Algorithm 1 can be adapted easily for attacking minimum weight problems with local failure criteria.*

8 Modified Algorithm

Algorithm 1, presented in Section 7 can find an optimal solution to the failure criterion design problem (3) in a finite number of steps, if such solution exists. This algorithm is adapted from the algorithm presented in [17] (refer to this article for details). In that article, it was proposed to replace the classical construction of the compliance GBD cuts based on displacements and Lagrange Multipliers related to 0 – 1 designs by similar ones, built from non necessarily 0 – 1 designs. These non necessarily 0 – 1 designs are to be found at the level set of the incumbent solution, and must be updated each time a better incumbent is found. The method supposes that the algorithm will easily find feasible

designs. Each time a better incumbent solution is found, the compliance GBD cuts are updated and improved, and the lower bound of the global optimal value provided by the GBD method is likely to be significantly improved, and more chances of convergence of the algorithm are expected. However, in local failure constrained problems, it might be extremely difficult to even find a single feasible design. In this case, the upper bound stays at a high value ($+\infty$) and the algorithm does not use the advantages of the algorithm using level set GBD cuts. To avoid this difficulty, we propose a strategy, which we motivate with the following illustrative example. Consider a 2D topology design problem specified by the design discretization and boundary conditions showed in Figure 8. Symmetry has been taken into account, which reduces the problem to a 24 design variables optimization problem. The material properties and the finite element modeling are the same as the one used in [20]. The stiffness rate material-void is set to $E_1/E_0 = 100$. We set a mass constraint of 68%. We first solved the minimum compliance problem without including any failure criterion, with an optimality tolerance of 0.5%, using the GBD algorithm proposed in [17]. The algorithm converged after solving 114 relaxed master problems, with an objective value of $c^* = 30.8153$. The solution of this problem is shown in Figure 2(a).

For each design obtained from the solution of each relaxed master problem, the stresses were evaluated (at the center of each design element), and the ∞ -norm of the stresses were compared, keeping the smallest value in the sequence. The value obtained was $\min_{1 \leq i \leq 114} \{\|\sigma_i\|_\infty\} = 2.2125$. On the other hand, the ∞ -norm of the stress for the solution of the minimum compliance problem (showed in Figure 2(a)) has a value of $\|\sigma^*\|_\infty = 3.767$. To test Algorithm 1, we included a maximum stress constraint not allowing the ∞ -norm of the stress σ to be larger than $\sigma_{max} = 2.22$, so we use a local failure function

$$F(x, u) = \|\sigma(x, u)\|_\infty - 2.22$$

for the optimization model (this value for σ_{max} ensures that the minimum compliance with maximum stress constraints problem is in fact feasible). The algorithm this time run for 233 iterations, obtaining an optimal design which is showed in Figure 2(b). The optimal solution found this time has compliance of $c_\sigma^* = 31.576$. For this example, the algorithm did not find a feasible design until 202 iterations were executed. This situation supposes that the main difficulty of this algorithm is to find a first feasible solution, so the inclusion of the level set cuts at the incumbent solution could have an impact in the performance of the algorithm. At the same time, it is very likely that a design satisfying the local failure constraints is to have a low value for the compliance (and otherwise the solution is not interesting). To avoid this problem, and to take advantage of the improvement of the performance of the algorithm when including these level set cuts, we propose a modified algorithm, where the level set considered to build the cuts are the ones from the current minimum compliance design (which maybe infeasible for the local failure constraint), while the convergence of the algorithm is measured with respect to the incumbent solution of the design problem. This idea supposes the existence of an extra artificial upper bound. This extra upper bound is used to build GBD cuts, and is equal to the objective value of the current minimum compliance found, regardless of the local failure criterion. The other bound is the upper bound of the objective value of the incumbent solution, and it is used to measure the convergence of the algorithm. For each design x^k obtained after solving a relaxed master problem, two GBD cuts, and eventually one or more feasibility cuts are included.

1. First, we include the corresponding U -level set GBD cut

$$-\{\nu_{l,U_c}^k\}^T \{u_{l,U_c}^k\} x - y \leq -2f_l^T \{u_{l,U_c}^k\}$$

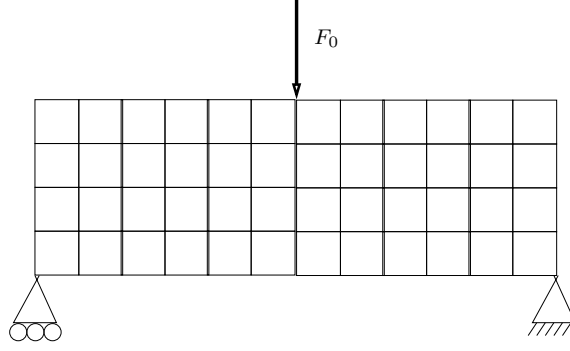


Figure 1: Example: Boundary conditions of an mbb beam.

(i.e., the one related to the design x_U^k).

2. The GBD cut with respect to x^k

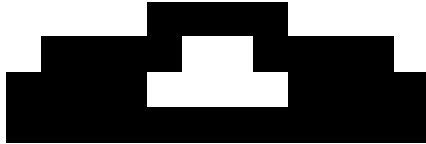
$$-\nu_l^{kT} u_l^k x - y \leq -2f_l^T u_l^k$$

3. Then, if the design is not feasible for the local failure criterion, a feasibility cut

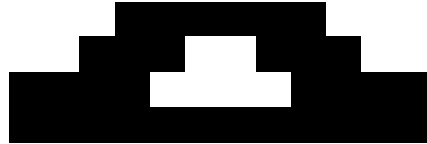
$$G_q(x) \leq 0, \quad q = 1, \dots, Q,$$

preventing x^k to be feasible in the next iterations are included.

Here we present the algorithm in a formal statement.



(a) No Failure Solution.



(b) Max Stress solution.

Algorithm 2: Generalized Benders' Decomposition for multiple load minimum compliance problem (3) with local failure criteria

1. [1A] (Optional Step) Run any heuristic procedure (if available) to obtain 0 – 1 designs for the optimal design problem. If P initial designs were found, number them in an arbitrary order x^1, \dots, x^P . If among these P designs found there are feasible designs for the optimization problem, set U as the objective of the incumbent solution (the design with best objective). Call this incumbent x^* . If c_1, c_2, \dots, c_P are the objectives for the sequence of designs, set $U_c = \min_{1 \leq k \leq P} \{c_k\}$.

[1B] (if [1A] is not executed) Set $P = 1$, the upper bound $U = U_c = +\infty$, Find an initial design $x^1 \in [0, 1]^{\tilde{n}}$ satisfying the mass constraint $\rho^T x^1 \leq M$, and the material selection constraints $\sum_{i=1}^{n^c} x_{ij} = 1$, $j = 1, \dots, n$.

2. Set $Q = 0$, the lower bound $y^* = -\infty$ and the convergence tolerance $\epsilon \geq 0$. Compute the displacement fields $u^k = u_l^k, \dots, u_m^k$, $u_l^k = K(x^k)^{-1} f_l$, the compliances c_l^k, \dots, c_m^k , $c_l^k = f_l^T u_l^k$, and the Lagrange multiplier vectors

$$\nu_l^k = ((K_{11} u_l^k)^T (K_{12} u_l^k)^T \dots (K_{n^c n} u_l^k)^T)^T,$$

for $l = 1, \dots, m, k = 1, \dots, P$. Notice that the initial design x^1 does not need to be a 0 – 1 design, and it does not need to satisfy the local failure criterion $F(x^1, u_l) \leq 0$, $l = 1, \dots, m$.

3. Solve the first relaxed master problem

$$\begin{aligned} & \underset{x \in \mathbb{R}^{\tilde{n}}, y \in \mathbb{R}}{\text{minimize}} && y \\ & \text{subject to} && -\nu_l^{kT} u_l^k x - y \leq -2f_l^T u_l^k, \quad l = 1, \dots, m, \quad k = 1, \dots, P, \\ & && \rho^T x \leq M, \\ & && \sum_{i=1}^{n^c} x_{ij} = 1, \quad j = 1, \dots, n \\ & && x \in \{0, 1\}^{\tilde{n}}, \end{aligned} \tag{M1}$$

by any solver for linear-mixed integer programming. If the problem is infeasible, stop and exit. The whole design problem is infeasible. Otherwise, if the problem is feasible, it necessarily has at least one optimal solution. This comes from the fact that the functions

$$h_l^k : \{0, 1\}^n \longrightarrow \mathbb{R}, \quad h_l^k(x) = -\nu_l^{kT} u_l^k x, \quad \forall k, l,$$

are bounded from below. Denote the solution of (M1) found by (x^{P+1}, y^{P+1}) . Set $P \rightarrow P + 1$.

4. Solve the equilibrium equations $K(x^P) u_l^P = f_l$, for $l = 1, \dots, m$, and compute the compliances $c_l^P = f_l^T u_l^P$, for $l = 1, \dots, m$.
5. For each $l = 1, \dots, m$, evaluate the equilibrium-local failure feasibility subproblem:

$$\begin{aligned} & \underset{u_1, \dots, u_m \in \mathbb{R}^d}{\text{minimize}} && \sum_{l=1}^m 0^T u_l \\ & \text{subject to} && K(x^P) u_l = f_l, \quad l = 1, \dots, m, \\ & && F(x^P, u_l) \leq 0. \end{aligned} \tag{57}$$

If the subproblem (57) is infeasible, compute one or more valid feasibility cuts preventing the design x^P from being feasible in the next master problem (this could be done, for example, by building GBD feasibility cuts after treating the reformulation of the feasibility system as a convex problem). Denote $G_{Q+1}(x) \leq d_{Q+1}, \dots, G_{Q+p}(x) \leq d_{Q+p}$ the found feasibility cuts. Set $Q \rightarrow Q + p$. Compute then the Lagrange multiplier vectors

$$\nu_l^P = ((K_{11} u_l^P)^T (K_{12} u_l^P)^T \dots (K_{n^c n} u_l^P)^T)^T, \quad \forall l = 1, \dots, m.$$

6. If (57) is feasible, and $\max_{1 \leq l \leq m} \{c_l^P\} < U$, then set $U = \max_{1 \leq l \leq m} \{c_l^P\}$, $x^* = x^P$.
7. If $\max_{1 \leq l \leq m} \{c_l^P\} < U_c$, then set $U_c = \max_{1 \leq l \leq m} \{c_l^P\}$ and update all designs, displacements and Lagrange multipliers to the level set of U_c (as is it explained in [17]), obtaining $x_{U_c}^k$, u_{l,U_c}^k , and ν_{l,U_c}^k , $k = 1, \dots, P$, $l = 1, \dots, m$. If (57) is feasible, and $\max_{1 \leq l \leq m} \{c_l^P\} \geq U_c$, compute the level set design $x_{U_c}^P$ (as it is explained in [17]), the corresponding displacements u_{l,U_c}^P , and Lagrange multipliers ν_{l,U_c}^P , $l = 1, \dots, m$.
8. If $U - y^* \leq \epsilon$, then stop. The optimal design found is x^* , with optimal value U . Otherwise, set $P \rightarrow P + 1$, and continue to step 9.
9. Solve the relaxed master problem

$$\begin{aligned}
& \underset{x \in \mathbb{R}^n, y \in \mathbb{R}}{\text{minimize}} && y \\
& \text{subject to} && -\nu_l^{kT} u_l^k x - y \leq -2f_l^T u_l^k, && k = 1, \dots, P, \quad l = 1, \dots, m, \\
& && -\{\nu_{l,U_c}^k\}^T \{u_{l,U_c}^k\} x - y \leq -2f_l^T \{u_{l,U_c}^k\}, && k = 1, \dots, P, \quad l = 1, \dots, m, \\
& && G_q(x) \leq 0, && q = 1, \dots, Q, \\
& && \rho^T x \leq M, \\
& && \sum_{i=1}^{n^c} x_{ij} = 1, && j = 1, \dots, n, \\
& && x \in \{0, 1\}^{\tilde{n}},
\end{aligned}$$

by any solver for linear-mixed 0 – 1 programming. If the relaxed master problem is infeasible, stop and exit. If a feasible design has been found, it is an optimal solution for the design problem. Otherwise, the design problem is infeasible. If the relaxed master problem is feasible and an optimal solution is found, denote the solution of this program by (x^{P+1}, y^{P+1}) . Set $P \rightarrow P + 1$. Return to step 4.

Remark 2. *Algorithm 2 can be adapted easily for solving minimum weight problems with local failure criteria.*

9 Local Failure feasibility Analysis

In this section, we study the feasibility of the system (55). As it was pointed out before, the system (55) can be reformulated as a convex system of constraints, given by (56). The equivalence of these formulations of the equilibrium-local failure feasibility system means that the feasibility of any of systems implies the feasibility of the other one. To simplify the notation, we consider in this section only a single load condition $f = f_1$, i.e., $m = 1$. Therefore, the index for the load case l is not used in any equation during this section.

Suppose that a given design x^k is infeasible for the equilibrium equations (55). Our goal is to find one or several linear constraints preventing the design x^k (and potentially other infeasible designs for (55)) to be feasible for the master problem. Suppose that the failure condition $F(x, u) \leq 0$ is linear in the displacement field u . Thus, we can apply the so called *combinatorial Benders' feasibility cuts* (see [5, 6, 10, 18, 24]) for the system (55). This idea was used already in [16]. In general, combinatorial Benders' cuts can be applied to linear systems, when these are inconsistent. The idea is to find a small subsystem of constraints, responsible for the infeasibility of the global system. These subsystems are called *Irreducible Inconsistent Subsystem* (SII), and imply a necessary relationship among the design variables. These relationships characterize the combinatorial Benders' cuts.

The Combinatorial Benders' feasibility cuts for the structural design problems studied in this article require the additional condition of linearity on the field u .

(A-7) The failure function $F(x, u)$ is linear in the displacement field u .

To generate this type of feasibility cuts, the feasibility system (55) is reformulated including additional variables $z_{ij} \in \mathbb{R}^d$, $i \in \{1, \dots, n^c\}$, $j \in \{1, \dots, n\}$, linking the displacement fields u and the design variable x_{ij} , as $z_{ij} = x_{ij}u$. The reformulation we consider for finding IIS's is given by

$$\sum_{i,j} K_{ij} z_{ij} = f, \quad (58a)$$

$$z_{ij} - u = 0, \quad \forall (i, j) \in \mathbb{N}_1(x), \quad (58b)$$

$$z_{ij} = 0, \quad \forall (i, j) \in \mathbb{N}_0(x), \quad (58c)$$

$$F(x, u) \leq 0. \quad (58d)$$

The index sets $\mathbb{N}_0(x)$, $\mathbb{N}_1(x)$ are given by

$$\begin{aligned} \mathbb{N}_0(x) &:= \{(i, j) : i \in \{1, \dots, n^c\}, j \in \{1, \dots, n\} \mid x_{ij} = 0\}, \\ \mathbb{N}_1(x) &:= \{(i, j) : i \in \{1, \dots, n^c\}, j \in \{1, \dots, n\} \mid x_{ij} = 1\}. \end{aligned} \quad (59)$$

The procedure to find IIS's is based on heuristics supported by dual optimization theory, and it was developed by several authors. See, for example [6, 10] for a deep understanding of the theory and heuristics involved. In [14], there is a more specific explanation about the heuristic procedure for obtaining IIS's for a version of the system (58), but without the non failure condition.

The other option to treat the Equilibrium feasibility system, is to generate GBD feasibility cuts to this system in its convex form, (56). Such cuts prevent the current design x^k (which is infeasible for the original mixed integer problem (3)) from being part of a feasible solution in the next master problems. However, There are a few necessary conditions to be able to build these cuts. In [14, 16] it was shown that a particular Lagrange duality result is necessary to be able to build Benders' feasibility constraints. Here we present a general result for convex failure criteria, which is a similar to Theorem 2.2 in [7], but before we write a result from the same article, which is necessary for the proof of the next theorem.

Theorem 1. *Consider the convex optimization problem*

$$\begin{aligned} &\underset{x \in X}{\text{minimize}} && f(x) \\ &\text{subject to} && g_i(x) \leq 0, \quad i = 1, \dots, m \end{aligned} \quad (60)$$

where each $g_i(x)$ is a convex function in the non empty convex set $X \in \mathbb{R}^n$. If $\{z \in \mathbb{R}^m : g_i(x) \leq z_i, i = 1, \dots, m, \text{ for some } x \in X\}$ is closed and the optimal value of the dual is finite, then the primal problem must be feasible.

Theorem 2. *A given design $x^k \in \{0, 1\}^{\bar{n}}$ satisfies is a solution of the feasibility problem (56), if and only if, it satisfies the infinite system*

$$l_*^{cr}(x^k, \eta, \Lambda, \phi) \leq 0, \quad \forall \eta \in \mathbb{R}^d, \Lambda \in \mathbb{R}_+^{n^\Lambda}, \phi \in \mathbb{R}^{n^\phi}, \quad (61)$$

where the function l_*^{cr} is defined as

$$l_*^{cr}(x^k, \eta, \Lambda, \phi) = \inf_{u, r, s} \{ \eta^T [Br - f] + \Lambda^T [A(r^{\min}, r^{\max})(x, u, r)^T] + \phi^T F(x^k, u, s) \}.$$

Proof. The proof is similar to the one of the Theorem 2.2 in [7]. Suppose that x^k is such that it exists (u^k, r^k, s^k) satisfying the feasibility system (56). It is straightforward to verify that (61) is satisfied as well. Now, suppose that (61) is satisfied. Then,

$$\sup_{\eta, \Lambda \geq 0, \phi \geq 0} \{l_*^{cr}(x^k, \eta, \Lambda, \phi)\} \leq 0, \quad (62)$$

It follows that, since the zero vector $(\eta_l^T = 0^T, \lambda^T = 0^T, \phi^T = 0^T)$ is allowed in (62), we have that

$$\sup_{\eta, \Lambda \geq 0, \phi \geq 0} \{l_*^{cr}(x^k, \eta, \Lambda, \phi)\} = 0. \quad (63)$$

This last equality implies that the dual of the feasibility problem (56) has an optimal value equal to 0. The feasible problem (56) can be written as a continuous vectorial constraint $G_{x^k}(u, r, s) \leq 0$. Therefore, the set $\{z \in \mathbb{R}^{n^G} : G_x(u, r, s) \leq z\}$ is closed (it could be expressed as $H^{-1}([-\infty, 0]^{n^G})$, where H is the continuous function $H(u, r, s, z) = G_x(u, r, s) - z$). Theorem 5.1 is verified for the convex feasibility problem (56). Therefore, the primal feasibility problem (56) must be feasible, and therefore, x^k is feasible for (56). \square

Theorem 2 implies that, whenever a design x^k is infeasible for the system (56), we can find the Lagrange multipliers $\eta^k \in \mathbb{R}^{n^\eta}$, $\Lambda^k = \{\lambda_i^k \in \mathbb{R}_+^{n_i}, i = 1, \dots, 4\}$, $\phi^k \in \mathbb{R}_+^{n^\phi}$, such that

$$l_*^{cr}(x^k, \eta^k, \Lambda^k, \phi^k) > 0.$$

Therefore, if the constraint (cut) $l_*^{cr}(x^k, \eta^k, \Lambda^k, \phi^k) \leq 0$ is included in the master problem, the feasible design x^k (and, hopefully, many other designs too) is prevented from being feasible in the following instances of the master problem.

9.1 Feasibility Problem for the Maximum Strain Criterion

The equilibrium-local failure feasibility condition for the maximum strain criterion (given by equation (3.1)), is given by

$$\begin{aligned} K(x^k)u &= f, \\ \sum_{i=1}^{n^c} d_{ijl} &\leq \sum_{i=1}^{n^c} x_{ij} \epsilon_{\max, i}, \quad \forall j, k, \\ -\sum_{i=1}^{n^c} d_{ijl} &\leq \sum_{i=1}^{n^c} x_{ij} \epsilon_{\max, i}, \quad \forall j, k, \\ d_{ijl} &= x_{ij} \Theta_i B_j^T u_l, \quad \forall i, j, k, \end{aligned} \quad (64)$$

Suppose that (64) is infeasible for a given design x^k .

The first possibility we have, is to use combinatorial Benders' cuts related to the system (64). The construction is again, similar to the one presented in [16], with the difference of the inclusion here of the maximum strain criterion in the system. The system considered for finding combinatorial Benders' cuts is given by

$$\begin{aligned}
\sum_{i,j} K_{ij} z_{ij} &= f, \\
z_{ij} - u &= 0, & \forall (i,j) \in \mathbb{N}_1(x^k), \\
z_{ij} &= 0, & \forall (i,j) \in \mathbb{N}_0(x^k), \\
\sum_{i=1}^{n^c} d_{ijl} &\leq \sum_{i=1}^{n^c} x_{ij} \epsilon_{\max,i}, & \forall j, k, \\
-\sum_{i=1}^{n^c} d_{ijl} &\leq \sum_{i=1}^{n^c} x_{ij} \epsilon_{\max,i}, & \forall j, k, \\
d_{ijl} &= x_{ij} \Theta_i B_j^T u_l, & \forall i, j, k,
\end{aligned} \tag{65}$$

The method is explained in detail in [16]. Refer to this article for details about the implementation of the algorithm obtaining combinatorial cuts.

The second possibility for feasibility constraints related to the equilibrium-max strain system, is to generate GBD feasibility cuts (see [14, 16, 12]). In order to do this, we need to consider the convex version of the equilibrium-max strain system.

$$\begin{aligned}
\underset{u \in \mathbb{R}^d, r, d}{\text{minimize}} \quad & c = \{0^T u + 0^T r\} \\
\text{subject to} \quad & Br = f, \\
& A(r^{\min}, r^{\max})(x, u, r)^T \leq 0, \\
& \sum_{i=1}^{n^c} d_j \leq \sum_{i=1}^{n^c} x_{ij} \epsilon_{\max,i}, & \forall j, \\
& -\sum_{i=1}^{n^c} d_j \leq \sum_{i=1}^{n^c} x_{ij} \epsilon_{\max,i}, & \forall j, \\
& x_{ij} d_j^{\min} \leq d_j & \forall i, j, \\
& d_j \leq x_{ij} d_j^{\max}, & \forall i, j, \\
& (1 - x_{ij}) d_j^{\min} \leq B_j^T u - d_j, & \forall i, j, \\
& B_j^T u - d_j \leq (1 - x_{ij}) d_j^{\max}, & \forall i, j.
\end{aligned} \tag{66}$$

Theorem 2, ensures that if x^k is an infeasible design for the equilibrium-local failure feasibility system (66), then it exists $\eta^k \in \mathbb{R}^{n^\eta}$, $\Lambda^k \in \mathbb{R}_+^{n^\Lambda}$, $\phi_i^k \in \mathbb{R}_+^{n^{\phi_i}}$, $i = 1, \dots, 6$ such that

$$l_*^s(x^k, \eta^k, \Lambda^k, \phi^k) > 0,$$

where l_*^s is defined as

$$\begin{aligned}
l_*^s(x^k, \eta^k, \Lambda^k, \phi^k) &:= \inf_{u, r, d} \{ \eta^{kT} [Br - f] + \Lambda^{kT} [A(r^{\min}, r^{\max})(x^k, u, r)^T] \\
&\quad + \phi_1^{kT} [\sum_{i=1}^{n^c} d_j - \sum_{i=1}^{n^c} x_{ij} \epsilon_{\max,i}] \\
&\quad + \phi_2^{kT} [-\sum_{i=1}^{n^c} d_j - \sum_{i=1}^{n^c} x_{ij} \epsilon_{\max,i}] \\
&\quad + \phi_3^{kT} [x_{ij} d_j^{\min} - d_j] \\
&\quad + \phi_4^{kT} [d_j - x_{ij} d_j^{\max}] \\
&\quad + \phi_5^{kT} [(1 - x_{ij}) d_j^{\min} - B_j^T u - d_j] \\
&\quad + \phi_6^{kT} [B_j^T u - d_j - (1 - x_{ij}) d_j^{\max}] \}.
\end{aligned}$$

Therefore, to prevent the infeasible design x^k from belonging to the feasible set of the master problem, we must include the constraint

$$l_*^s(x^k, \eta^k, \Lambda^k, \phi^k) \leq 0,$$

in the following relaxed master problem.

9.2 Feasibility Problem for the Maximum Stress Criterion

The failure feasibility condition for the maximum strain criterion (given by equation (20)), is given by

$$\begin{aligned} K(x^k)u &= f, \\ \sum_{i=1}^{n^c} C_i x_{ij}^k \Theta_i B_j^T u &\leq \sum_{i=1}^{n^c} x_{ij} \sigma_{\max,i}, \quad j = 1, \dots, n, \\ -\sum_{i=1}^{n^c} C_i x_{ij}^k \Theta_i B_j^T u &\leq \sum_{i=1}^{n^c} x_{ij} \sigma_{\max,i}, \quad j = 1, \dots, n, \end{aligned} \quad (67)$$

Suppose that (67) is infeasible for a given design x^k . The first possibility we have is to use combinatorial Benders' cuts related to the system (67). The construction is again, similar to the one presented in [16], with the difference of the inclusion of the maximum stress criterion in the system. The system considered for finding combinatorial Benders' cuts is given by

$$\begin{aligned} \sum_{i,j} K_{ij} z_{ij} &= f, \\ z_{ij} - u &= 0, & \forall (i,j) \in \mathbb{N}_1(x), \\ z_{ij} &= 0, & \forall (i,j) \in \mathbb{N}_0(x), \\ \sum_{i=1}^{n^c} C_i x_{ij} \Theta_i B_j^T u &\leq \sum_{i=1}^{n^c} x_{ij} \sigma_{\max,i}, \quad j = 1, \dots, n, \\ -\sum_{i=1}^{n^c} C_i x_{ij} \Theta_i B_j^T u &\leq \sum_{i=1}^{n^c} x_{ij} \sigma_{\max,i}, \quad j = 1, \dots, n, \end{aligned} \quad (68)$$

The method was explained in detail in [16]. Refer to this article for details about the implementation of the algorithm obtaining combinatorial cuts.

The second possibility for feasibility constraints related to the equilibrium-local failure system, is to generate GBD feasibility cuts (see [14, 16, 12]). In order to do this, we need to consider the convex version of the equilibrium-local failure system for the maximum stress criterion.

$$\begin{aligned} \underset{u \in \mathbb{R}^d, r, d}{\text{minimize}} \quad & c = \{0^T u + 0^T r\} \\ \text{subject to} \quad & Br = f, \\ & A(r^{\min}, r^{\max})(x, u, r)^T \leq 0, \\ & \sum_{i=1}^{n^c} C_i d_j \leq \sum_{i=1}^{n^c} x_{ij} \sigma_{\max,i}, \quad \forall j, \\ & -\sum_{i=1}^{n^c} C_i d_j \leq \sum_{i=1}^{n^c} x_{ij} \sigma_{\max,i}, \quad \forall j, \\ & x_{ij} d_j^{\min} \leq d_j, \quad \forall i, j, \\ & d_j \leq x_{ij} d_j^{\max}, \quad \forall i, j, \\ & (1 - x_{ij}) d_j^{\min} \leq B_j^T u - d_j, \quad \forall i, j, \\ & B_j^T u - d_j \leq (1 - x_{ij}) d_j^{\max}, \quad \forall i, j. \end{aligned} \quad (69)$$

Theorem 2, ensures that if x^k is an infeasible design for the equilibrium-local failure feasibility system (69), then it exists $\eta^k \in \mathbb{R}^{n^\eta}$, $\Lambda^k \in \mathbb{R}_+^{n^\Lambda}$, $\phi_i^k \in \mathbb{R}_+^{n^{\phi_i}}$, $i = 1, \dots, 6$ such that

$$l_*^{ss}(x^k, \eta^k, \Lambda^k, \phi^k) > 0,$$

where l_*^{ss} is defined as

$$\begin{aligned} l_*^{ss}(x^k, \eta^k, \Lambda^k, \phi^k) := & \inf_{u, r, d} \{ \eta^{kT} [Br - f] + \Lambda^{kT} [A(r^{\min}, r^{\max})(x^k, u, r)^T] \\ & + \phi_1^{kT} [\sum_{i=1}^{n^c} C_i d_j - \sum_{i=1}^{n^c} x_{ij} \sigma_{\max, i}] \\ & + \phi_2^{kT} [-\sum_{i=1}^{n^c} C_i d_j - \sum_{i=1}^{n^c} x_{ij} \sigma_{\max, i}] \\ & + \phi_3^{kT} [x_{ij} d_j^{\min} - d_j] \\ & + \phi_4^{kT} [d_j - x_{ij} d_j^{\max}] \\ & + \phi_5^{kT} [(1 - x_{ij}) d_j^{\min} - B_j^T u - d_j] \\ & + \phi_6^{kT} [B_j^T u - d_j - (1 - x_{ij}) d_j^{\max}]. \end{aligned}$$

Therefore, to prevent the infeasible design x^k from belonging to the feasible set of the master problem, we must include the constraint

$$l_*^{ss}(x^k, \eta^k, \Lambda^k, \phi^k) \leq 0,$$

in the following master problems.

9.3 Feasibility Problem for the Tsai-Hill Criterion

Here we consider the Tsai-Hill failure criterion. The failure feasibility system for this failure ((33)) is represented explicitly by

$$\begin{aligned} Br &= f, \\ A(r^{\min}, r^{\max}) (x, u, r)^T &\leq 0, \\ \sum_{i=1}^k s_{ij}^T s_{ij} &\leq 1, & \forall j, \\ x_{ij}^k s_{ij}^{\min} - s_{ij} &\leq 0, & \forall i, j, \\ s_{ij} - x_{ij}^k s_{ij}^{\max} &\leq 0, & \forall i, j, \\ -x_{ij}^k s_{ij}^{\min} - B_{ij}^T u + s_{ij} &\leq -s_{ij}^{\min}, & \forall i, j, \\ B_{ij}^T u + x_{ij}^k s_{ij}^{\max} - s_{ij} &\leq s_{ij}^{\max}, & \forall i, j. \end{aligned} \tag{70}$$

In order to generate Benders' feasibility cut (see [14, 16, 12]), we need to consider the convex version of the equilibrium-local failure system for the maximum strain criterion.

Suppose that (70) is infeasible for a given design x^k . Theorem 2, ensures that if x^k is an infeasible design for the equilibrium-local failure feasibility system (70), then it exists $\eta^k \in \mathbb{R}^{n^\eta}$, $\Lambda^k \in \mathbb{R}_+^{n^\Lambda}$, $\phi_i^k \in \mathbb{R}_+^{n^{\phi_i}}$, $i = 1, \dots, 5$, such that

$$l_*^{th}(x^k, \eta^k, \Lambda^k, \phi^k) > 0,$$

where l_*^{th} is defined as

$$\begin{aligned}
l_*^{th}(x^k, \eta^k, \Lambda^k, \phi^k) := & \inf_{u, r, d} \{ \eta^{kT} [Br - f] \\
& + \Lambda^{kT} [A(r^{\min}, r^{\max})(x^k, u, r)^T] \\
& + \phi_1^{kT} [\sum_{i=1}^k s_{ij}^T s_{ij} - 1] + \phi_2^{kT} [x^k i j s_{ij}^{\min} - s_{ij}] \\
& + \phi_3^{kT} [s_{ij} - x_{ij}^k s_{ij}^{\max}] \\
& + \phi_4^{kT} [-x_{ij}^k s_{ij}^{\min} - B_{ij}^T u + s_{ij} + s_{ij}^{\min}] \\
& + \phi_5^{kT} [B_{ij}^T u + x_{ij}^k s_{ij}^{\max} - s_{ij} - s_{ij}^{\max}] \}.
\end{aligned}$$

Therefore, to prevent the infeasible design x^k from belonging to the feasible set of the master problem, we must include the constraint

$$l_*^{th}(x^k, \eta^k, \Lambda^k, \phi^k) \leq 0,$$

in the relaxed master problem of the next iteration.

As additional remark, we point out that the system (70) is not a linear system when fixing the variable x^k . Therefore it is not possible to obtain combinatorial Benders' for this type of system.

9.4 Feasibility Problem for the Tsai-Wu Criterion

Here we consider the Tsai-Wu failure criterion. The failure feasibility system for this failure ((39)) is represented explicitly by

$$\begin{aligned}
K(x^k)u &= f, \\
\sum_{i=1}^k s_{ij}^T s_{ij} + \eta_{ij} &\leq 1, & \forall j, \\
-s_{ij} + x_{ij}^k s_{ij}^{\min} &\leq 0, & \forall i, j, \\
s_{ij} - x_{ij}^k s_{ij}^{\max} &\leq 0, & \forall i, j, \\
-B_{ij}^T u + s_{ij} - x_{ij}^k s_{ij}^{\min} &\leq -s_{ij}^{\min}, & \forall i, j, \\
B_{ij}^T u - s_{ij} + x_{ij}^k s_{ij}^{\max} &\leq s_{ij}^{\max}, & \forall i, j, \\
-\eta_{ij} + x_{ij}^k \eta_{ij}^{\min} &\leq 0, & \forall i, j, \\
\eta_{ij} - x_{ij}^k \eta_{ij}^{\max} &\leq 0, & \forall i, j, \\
-h^T C_i \Theta_i B_j^T u + \eta_{ij} - x_{ij}^k \eta_{ij}^{\min} &\leq -\eta_{ij}^{\min} & \forall i, j, \\
h^T C_i \Theta_i B_j^T u - \eta_{ij} + x_{ij}^k \eta_{ij}^{\max} &\leq \eta_{ij}^{\max} & \forall i, j.
\end{aligned} \tag{71}$$

Suppose that (71) is infeasible for a given design x^k .

Theorem 2, ensures that if x^k is an infeasible design for the equilibrium-local failure feasibility system (71), then it exists $\eta^k \in \mathbb{R}^{n_\eta}$, $\Lambda^k \in \mathbb{R}_+^{n_\Lambda}$, $\phi_i^k \in \mathbb{R}_+^{n_{\phi_i}}$, $i = 1, \dots, 9$ such that

$$l_*^{tw}(x^k, \eta^k, \Lambda^k, \phi^k) > 0,$$

where l_*^{tw} is defined as

$$\begin{aligned}
l_*^{tw}(x^k, \eta^k, \Lambda^k, \phi^k) := & \inf_{u, r, d} \{ \eta^{kT} [Br - f] + \Lambda^{kT} [A(r^{\min}, r^{\max})(x^k, u, r)^T] \\
& + \phi_1^{kT} [\sum_{i=1}^k s_{ij}^T s_{ij} + \eta_{ij} - 1] \\
& + \phi_2^{kT} [x_{ij}^k s_{ij}^{\min} - s_{ij}] \\
& + \phi_3^{kT} [s_{ij} - x_{ij}^k s_{ij}^{\max}] \\
& + \phi_4^{kT} [-x_{ij}^k s_{ij}^{\min} - B_{ij}^T u + s_{ij} + s_{ij}^{\min}] \\
& + \phi_5^{kT} [B_{ij}^T u + x_{ij}^k s_{ij}^{\max} - s_{ij} - s_{ij}^{\max}] \\
& + \phi_6^{kT} [-\eta_{ij} + x_{ij}^k \eta_{ij}^{\min}] \\
& + \phi_7^{kT} [\eta_{ij} - x_{ij}^k \eta_{ij}^{\max}] \\
& + \phi_8^{kT} [-h^T C_i \Theta_i B_j^T u + \eta_{ij} - x_{ij}^k \eta_{ij}^{\min} + \eta_{ij}^{\min}] \\
& + \phi_9^{kT} [h^T C_i \Theta_i B_j^T u - \eta_{ij} + x_{ij}^k \eta_{ij}^{\max} - \eta_{ij}^{\max}].
\end{aligned}$$

Therefore, to prevent the infeasible design x^k from belonging to the feasible set of the master problem, we must include the constraint

$$l_*^{tw}(x^k, \eta^k, \Lambda^k, \phi^k) \leq 0,$$

in the following master problems. Notice that the system (71) is not a linear system when fixing the variable x^k . Therefore, Combinatorial Benders' cuts do not either apply for the Tsai-Wu failure criterion.

9.5 General Local Failure Feasibility Cuts

In this section, we describe the implementation of a general type of feasibility cuts, which is valid for any type failure criterion function, independently of the mathematical properties, such as convexity, as it was the case in previous sections. Consider the failure feasibility system (55), where the function $F(x, u)$ does not necessarily satisfy any convexity assumption. We neither require any differentiability nor continuity property. The only assumption we suppose is that the function $F(x, u)$ has a finite value in the set

$$\{x \in \{0, 1\}^{\tilde{n}}, \sum_{j=1}^{n^c} x_{ij} = 1, u = u(x)\}$$

where $u(x)$ represents the unique solution to the equilibrium equations (1).

The idea of this feasibility analysis is the following. For each instance of the feasibility system (55), for which the couple $(x^k, u(x^k))$ is infeasible, a weak cut $C_w(x^k)$ preventing the design x^k from being feasible is created. The cut $C_w(x^k)$ does not prevent any other design in $\{0, 1\}^{\tilde{n}}$ from being feasible. Therefore, the validity of this cut is not under discussion, no matter which failure function we are considering.

This cut is created in the following way. Consider the design $x^k \in \{0, 1\}^{\tilde{n}}$, which is infeasible for the feasibility system (55). The cut $C_w(x^k)$ is defined as

$$C_w(x^k) : c_w^T x \leq b_w, \quad (72)$$

where $c_w \in \mathbb{R}^{\tilde{n}}$ is defined as:

$$c_w(i) = \begin{cases} -1, & \text{if } x(i) = 1, \quad i = 1, \dots, \tilde{n}, \\ 0, & \text{if } x(i) = 0, \quad i = 1, \dots, \tilde{n}. \end{cases} \quad (73)$$

$$b_w = 1 - \sum_{i=1}^{\tilde{n}} x(i)$$

Is it important to note that the feasibility cut $C_w(x^k)$, given by (72) is the weakest possible cut. It is equivalent to a Combinatorial Benders' cut (see [5, 6, 10, 18, 24]) of a linear system of equations with no information about inconsistent sub systems. This lack of information is the price we pay in order to be able to treat the general type of failure criteria. Its validity of this type of cut is based on the convexity of the \tilde{n} -dimensional cube $[0, 1]^{\tilde{n}}$, where each design correspond to one corner of the cube. It is therefore possible to impose a linear constraint removing a specific corner of the cube from the feasible set of the design problem, without removing any other corner of the cube. The principle of these cuts is in fact quite simple and could be understood in the following way. If a given design vector x^k is infeasible for the criterion $F(x, u) \leq 0$, then we know that we need to change the value of at least one component of the design vector x^k to have the possibility of finding a feasible design for the considered failure criterion. It is possible to argue that this type of cuts is too weak to be implemented in an optimization design problem. We do not deny this as a possible fact, but this should be tested in numerical experiments to see whether it is the case or not.

10 Implementation

The Generalized Benders' Decomposition method presented in this article is implemented in the numerical environment and academic software MUST ([1]). This software is specialized for optimal design of composite laminated structures and shell finite element analysis. The solver used in the treatment of the master problem is the commercial branch-and-cut solver for integer programming *Gurobi 3.0*, [2]. The continuous relaxations (4) and (6) are solved using the sequential quadratic programming package SNOPT, see [8, 9].

11 Numerical Examples

In this section, we present a preliminary numerical example for the Generalized Benders' Decomposition method proposed in this article. The example showed in this article corresponds to the problem of designing an optimal fiber angle selection of a 2-D squared membrane. The structure is clamped along the edge of one side, it is free along all other sides, and that is subject to a load in the y -direction, as it shown in Figure 2(c). The design domain is discretized in 20 by 20 finite elements for the analysis, using standard 9-nodes 2-D elements. At the same time, it is divided in 10 by 10 design elements. This difference in the discretization means that 4 finite elements are linked in a single design element. The algorithm was implemented in the platform for the analysis and optimization of shell, plates, and layered structures MUST ([1])

We start by solving the minimum compliance problem (3) without including any local failure in the formulation. The objective is to obtain a reference in the values of the strain of the optimal solution found. For this example, The GBD algorithm for problem (3) failure-unconstrained, stopped after 1210 iterations, employed a CPU-time of 7.4[h], obtaining a final optimality gap of 2.6%. The solution obtained is showed in Figure 2(d).

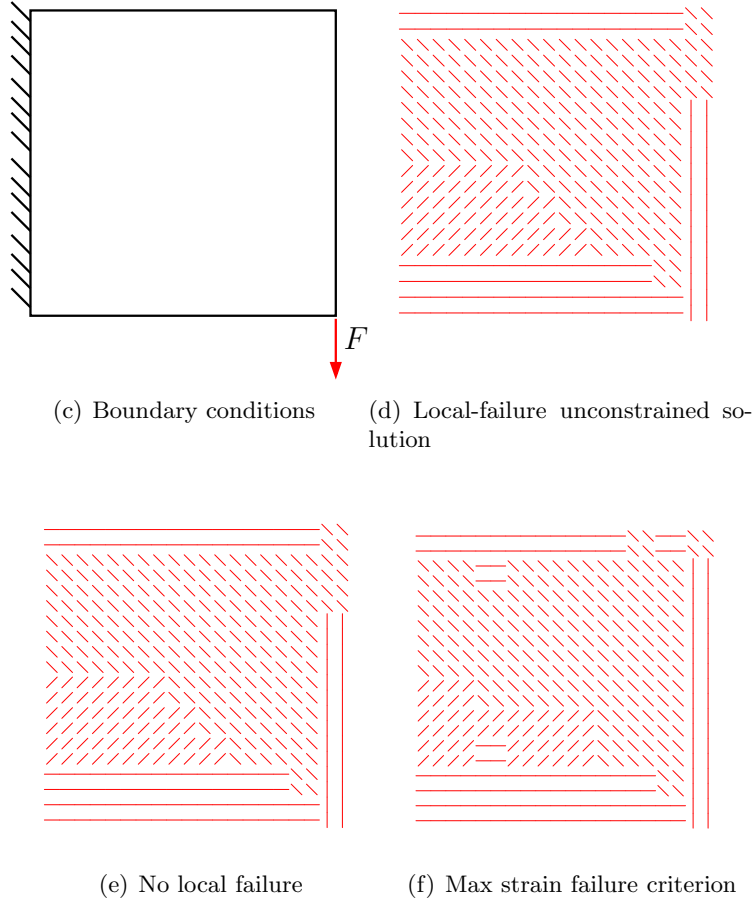


Figure 2: Design domain and ground structure considered in the numerical examples.

Its value for the maximum strain criterion is $\max\{\epsilon\} = 5.397 \cdot 10^{-3}$. To test the GBD algorithm (Algorithm 2 in section 8) in a local failure minimum compliance problem, we consider Problem (3), with a local failure condition $F(x, u) = \|\epsilon(x, u)\| - \epsilon_{\max}$, the maximum strain criterion. We set $\epsilon_{\max} = 5.396 \cdot 10^{-3}$, to ensure that the optimal design of the local failure unconstrained problem is infeasible for the maximum strain minimum compliance problem. This time, the algorithm run for 1240 iterations, employed a CPU-time of 8.5[h], and stopped finding a design with a optimality gap of 2.87%. The maximum strain value for this example is $5.318 \cdot 10^{-3}$ confirming the feasibility of the design for problem (P).

In Figures 2(e), 2(f) we can see the comparison between the solutions obtained when solving the unconstrained local minimum minimum compliance problem, and the maximum strain criterion-minimum compliance problem. These two design have 8 design variables with a different value.

12 Final Remarks and Future Work

We have successfully implemented a Generalized Benders' Decomposition algorithm which can solve minimum compliance problems considering any local failure criteria to global optimality. The method can be easily extended to minimum weight problem with local failure criteria. To our knowledge, this is the first serious investigation in the field of Structural Optimization considering solving structural design problems with local failure

criteria to global optimality. We have tested the proposed algorithm on a medium size angle selection design problem related to a single layered square plate obtaining convergence within a small tolerance. We have shown the formulation of several local failure formulations to be included in the standard minimum compliance and minimum weight problems. It has been showed that for some specific failure criteria, a specific convex reformulation can be used to implement GBD feasibility cuts which are valid in the sense of respecting the convergence to global optimality of the algorithm. In addition, an alternative and general implementation of feasibility cuts for general failure criteria, independent of mathematical properties, is proposed to avoid the resolution of the feasibility subproblem, and to be able to attack any type of local failure problem. This implementation trades the weakness of the feasibility cuts, but gains in the flexibility of the algorithm with respect to the use of local failure criteria. It has also been shown the particular implementation of the GBD algorithm to attack structural problems with high efficiency. For the future, the test of the algorithm with a set of examples, in order to evaluate the performance and robustness of the method in different design problems is necessary, specially considering the promising preliminary results obtained. The implementation of the GBD feasibility cuts for the convex reformulation of the feasibility problem must be carried out. In general a broad study considering a larger set of local failure criteria must be initiated.

References

- [1] MUST website, 2008. <http://www.mee.aau.dk/MUST/>.
- [2] GUROBI website, 2010. <http://www.gurobi.com/>.
- [3] J.F. Benders. Partitioning procedures for solving mixed-variables programming problems. *Numerische Mathematik*, 4(1):238–252, 1962.
- [4] M.P. Bendsøe and O. Sigmund. *Topology Optimization: Theory, Methods and Applications*. Springer-Verlag, 2nd edition, 2003.
- [5] N. Chakravarti. Some results concerning post-infeasibility analysis. *European Journal of Operational Research*, (73):139–143, 1994.
- [6] G. Codato and M. Fischetti. Combinatorial Benders’ cuts for mixed-integer linear programming. *Operations Research*, 54(4):756–766, 2006.
- [7] A.M. Geoffrion. Generalized Benders decomposition. *Journal of Optimization Theory and Applications*, 10(4):237–260, 1972.
- [8] P.E. Gill, W. Murray, and M.A. Saunders. Users guide for SNOPT 7.1: A Fortran package for large-scale nonlinear programming. *Numerical Analysis Report NA 04-1*, 2004.
- [9] P.E. Gill, W. Murray, and M.A. Saunders. SNOPT: An SQP algorithm for large-scale constrained optimization. *SIAM Journal on Optimization*, 47(1):99–131, 2005.
- [10] J. Gleeson and J. Ryan. Identifying minimal infeasible subsystems of inequalities. *ORSA Journal on Computing*, 2(1):61–63, 1990.
- [11] R.M. Jones. *Mechanics of Composite Materials*. Taylor and Francis, Inc., 2nd edition, 1999.

- [12] R. Lazimy. Extension of the generalized Benders decomposition. *Communications in Applied Numerical Methods*, 6:195–203, 1986.
- [13] W.P. Lin and H.T. Hu. Parametric study on the failure of fiber-reinforced composite laminates under biaxial tensile load. *Journal of Composite Materials*, 36(12):1481, 2002.
- [14] E. Munoz. *Global Optimization for Structural Design by Generalized Benders' Decomposition*. PhD thesis, Department of Mathematics, Technical University of Denmark, 2010. Chapter 3.
- [15] E. Munoz. *Global Optimization for Structural Design by Generalized Benders' Decomposition*. PhD thesis, Department of Mathematics, Technical University of Denmark, 2010. Chapter 6.
- [16] E. Munoz. *Global Optimization for Structural Design by Generalized Benders' Decomposition*. PhD thesis, Department of Mathematics, Technical University of Denmark, 2010. Chapter 4.
- [17] E. Munoz. *Global Optimization for Structural Design by Generalized Benders' Decomposition*. PhD thesis, Department of Mathematics, Technical University of Denmark, 2010. Chapter 5.
- [18] M. Parker and J. Ryan. Finding the minimum weight IIS cover of an infeasible system of linear inequalities. *Annals of Mathematics and Artificial Intelligence*, 17(1-2):107–126, 1996.
- [19] M.E. Plesha R. Cook, D. Malkus and R.J. Witt. *Concepts and Applications of Finite element Analysis*. John Wiley and Sons, 4th edition edition, 2002.
- [20] O. Sigmund. A 99 line topology optimization code written in Matlab. *Structural and Multidisciplinary Optimization*, 21(2):120–127, 2001.
- [21] J. Stegmann and E. Lund. Discrete material optimization of general composite shell structures. *International Journal for Numerical Methods in Engineering*, 62(14):2009–2027, 2005.
- [22] M. Stolpe and K. Svanberg. Modelling topology optimization problems as linear mixed 0–1 programs. *International Journal for Numerical Methods in Engineering*, 57:723–739, 2003.
- [23] S.W. Tsai. *Theory of composites design*. Dayton, 1992.
- [24] J.N.M. van Loon. Irreducibly inconsistent systems of linear inequalities. *European Journal of Operational Research*, 8(3):283–288, 1981.

Concluding Remarks

This thesis provides a large amount of new knowledge to the scientific community, in fundamental research, as well as in applied engineering sciences. The Thesis develops a new technique, based in the Generalized Benders' Decomposition method, to obtain global optimal solution of structural design problems. This new technique changes fundamentally the philosophy of its predecessor, and improves dramatically its performance. Besides, several heuristic procedures to improve the performance of the algorithm have been implemented. In overall, the resulting technique seems to be competitive with the most efficient technique for Mixed Integer Optimization existing nowadays. It has been successfully applied in the design of truss structures, topology optimization problems, and in multimaterial composite laminated structures. Even though the purpose of this thesis was to develop methods for Structural Optimization, the impact of this new technique can easily go to the whole mixed integer optimization field. It represents, to our knowledge, the first serious work in Global Optimization for structural design problems, which includes local failure conditions in the design problem.

8.1 Future Work

The results delivered by this thesis opens several investigation lines to the future, considering different levels of applicability. In the more fundamental area, the development of a new method for global optimization in structural design problems conjectures the existence of a new method for Global Optimization in the entire area of Mixed Integer Optimization. The results are promising, and everything indicates that the all results in this thesis can be extended to general mixed integer problems with convex continuous relaxations. At the same time, the generalization of this new technique for general types of problems implies that a corresponding benchmarking with respect to the most efficient techniques in Mixed Integer Optimization must be carried out in the close future. At a more applied level, the numerical study of local failure criteria in structural optimization problems must be continued and extended. Since it does not exist similar results in the field, a large number of investigation should be carried out. This means that serious studies considering each specific failure criterion being relevant in the design of composite laminated structures. Of course the study can be extended to the design of other types of structures, and considering other types of analysis modeling, and physics involved. In order to apply the method for coarse finite element discretizations, the algorithm must be optimized in the sense of minimizing the number of function evaluations,

which would be eventually the problem of the method as it is right now. This could be done efficiently by the introduction of an optimized line search algorithm in the level set design search procedure of the algorithm. This change may have consequences in real life size design problems. Besides, it is important to note that we have developed a global optimization method, which uses a different discretization for the analysis problem (FEM discretization) and for the design problem (design discretization). Since the present limitations of the method are related to the design discretization, and not for the analysis discretization, the method can certainly be applied for the design of real sized analysis discretization structures, as long as the design discretization remained coarse enough. This implies that we could use the method to optimize real size analysis models, discretized in a very fine mesh of finite elements (of order of millions of elements), as long as we use a coarse design discretization, using no more than 50.000 design variables (this is in fact not too ambitious, since in this thesis, we were able to solve to optimality with a small tolerance a design problem of 23.000 design variables).

